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Determinant and Inverse of a Skew Peoeplitz Matrix$^\sharp$

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Authors' contributions

This work was carried out in collaboration between both authors. Author ZL designed the study and proposed the concerned problem. Author MH performed the statistical analysis and wrote the first draft of the manuscript. Both authors read and approved the final manuscript.

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Abstract

In this paper, we consider the determinant and the inverse of a skew Peoeplitz matrix and a skew Peankel matrix involving Perrin numbers. we first give the definition of a skew Peoeplitz matrix and a skew Peankel matrix. Then we compute the determinant and the inverse of the skew Peoeplitz matrix and the skew Peankel matrix by constructing the transformation matrices.

Keywords: Skew Peoeplitz matrix; Perrin sequence; skew Peankel matrix; determinant; inverse.

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1 Introduction

The Perrin sequences are defined by the following recurrence relations [1, 2], respectively:

$$R_n = R_{n-2} + R_{n-3},$$

where $R_0 = 3, R_1 = 0, R_2 = 2$, $n \geq 3$. Let r_1, r_2 and r_3 be the roots of the characteristic equation $x^3 - x - 1 = 0$, then we have

$$\begin{cases} r_1 + r_2 + r_3 = 0, \\ r_1 r_2 + r_1 r_3 + r_2 r_3 = -1, \\ r_1 r_2 r_3 = 1. \end{cases}$$

The Binet formulas of the Perrin sequences $\{R_n\}$ have the form

$$R_n = r_1^n + r_2^n + r_3^n,$$

It is an ideal research area and hot topic for the inverses of the special matrices with famous numbers. Some scholars showed the explicit determinants and inverses of the special matrices involving famous numbers. Circulant matrices with Fibonacci and Lucas numbers are discussed and their explicit determinants and inverses are proposed in [3]. The authors provided determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [4]. Furthermore, in [5] the determinants and inverses are discussed and evaluated for Tribonacci skew circulant type matrices. The determinants and inverses of Tribonacci circulant type matrices are discussed in [6]. The explicit determinants of circulant and left circulant matrices including Tribonacci numbers and generalized Lucas numbers are shown based on Tribonacci numbers and generalized Lucas numbers in [7]. In [8], circulant type matrices with the k-Fibonacci and k-Lucas numbers are considered and the explicit determinants and inverse matrices are presented by constructing the transformation matrices. Jiang et al. [9] gave the invertibility of circulant type matrices with the sum and product of Fibonacci and Lucas numbers and provided the determinants and the inverses of these matrices. In [10], Jiang and Hong gave the exact determinants of the RSFPLR circulant matrices and the RSLPFL circulant matrices involving Padovan, Perrin, Tribonacci and the generalized Lucas numbers by the inverse factorization of polynomial. It should be noted that Jiang and Zhou [11] obtained the explicit formula for spectral norm of an r-circulant matrix whose entries in the first row are alternately positive and negative, and the authors [12] investigated explicit formulas of spectral norms for g-circulant matrices with Fibonacci and Lucas numbers. The authors [13] proposed the invertibility of the generalized Lucas skew circulant type matrices and provided their determinants and the inverse matrices. Sun and Jiang [14] gave the determinant and inverse of the complex Fibonacci Hermitian Toeplitz matrix by constructing the transformation matrices. Determinants and inverses of Fibonacci and Lucas skew symmetric Toeplitz matrices are given by constructing the special transformation matrices in [15].

In this paper we adopt the following two conventions $0^0 = 1$, $\mathrm{i}^2 = -1$, and we define two kinds of special matrix as follows.

Definition 1.1. A skew Peoeplitz matrix is a square matrix of the form

$$\mathbf{T}_{R,n} = \begin{pmatrix} R_0 & R_1 & R_2 & \cdots & R_{n-3} & R_{n-2} & R_{n-1} \\ -R_1 & R_0 & R_1 & \ddots & R_{n-4} & R_{n-3} & R_{n-2} \\ -R_2 & -R_1 & R_0 & \ddots & \ddots & R_{n-4} & R_{n-3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -R_{n-3} & -R_{n-4} & \ddots & \ddots & R_0 & R_1 & R_2 \\ -R_{n-2} & -R_{n-3} & -R_{n-4} & \ddots & -R_1 & R_0 & R_1 \\ -R_{n-1} & -R_{n-2} & -R_{n-3} & \cdots & -R_2 & -R_1 & R_0 \end{pmatrix}_{n\times n}, \tag{1.1}$$

where $R_i(0 \le i \le n-1)$ are the Perrin numbers, skew Peoeplitz matrix is Toeplitz matrix with Perrin number as its entries.

Definition 1.2. A skew Peankel matrix is a square matrix of the form

$$\mathbf{H}_{R,n} = \begin{pmatrix} R_{n-1} & R_{n-2} & R_{n-3} & \cdots & R_2 & R_1 & R_0 \\ R_{n-2} & R_{n-3} & R_{n-4} & \iddots & R_1 & R_0 & -R_1 \\ R_{n-3} & R_{n-4} & \iddots & \iddots & R_0 & -R_1 & -R_2 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ R_2 & R_1 & R_0 & \iddots & \iddots & -R_{n-4} & -R_{n-3} \\ R_1 & R_0 & -R_1 & \iddots & -R_{n-4} & -R_{n-3} & -R_{n-2} \\ R_0 & -R_1 & -R_2 & \cdots & -R_{n-3} & -R_{n-2} & -R_{n-1} \end{pmatrix}_{n\times n}, \tag{1.2}$$

where $R_i(0 \le i \le n-1)$ are the Perrin numbers, skew Peankel matrix is Hankel matrix with Perrin number as its entries.

It is easy to check that

$$\mathbf{H}_{R,n} = \mathbf{T}_{R,n}\hat{I}_n \ , \tag{1.3}$$

Let $\hat{\mathbf{I}}_n$ be the slip matrix of order n, i.e.

$$\hat{\mathbf{I}}_n = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}_{n\times n},$$

which provides us with basic relations between $\mathbf{T}_{R,n}$ and $\mathbf{H}_{R,n}$.

Lemma 1.1. *([16, 17]) A Toeplitz-Hessenberg matrix is an $n \times n$ matrix of the form*

$$T_H\left(\begin{array}{cccc}\kappa_0, & \kappa_1, & \cdots, & \kappa_n \\ \kappa_0, & \kappa_1, & \cdots, & \kappa_n\end{array}\right) = \begin{pmatrix} \kappa_1 & \kappa_0 & 0 & \cdots & \cdots & 0 \\ \kappa_2 & \kappa_1 & \kappa_0 & \ddots & & \vdots \\ \kappa_3 & \kappa_2 & \kappa_1 & \kappa_0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \kappa_{n-1} & \cdots & \cdots & \kappa_2 & \kappa_1 & \kappa_0 \\ \kappa_n & \cdots & \cdots & \cdots & \kappa_2 & \kappa_1 \end{pmatrix}_{n\times n},$$

where $\kappa_0 \neq 0$ and $\kappa_i \neq 0$ for at least one $i > 0$, the inverse of T_H is given by

$$T_H^{-1} = A = \begin{pmatrix} A_{1k} \\ A_{2k} \\ A_{3k} \\ \vdots \\ A_{n-1k} \\ A_{nk} \end{pmatrix}_{n\times 1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & \cdots & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \cdots & a_{nn} \end{pmatrix}_{n\times n},$$

where

$$\begin{cases} A_{1k} = \dfrac{-\sum\limits_{i=k+1}^{n} f_{ik}\kappa_{n-i+1}}{\kappa_n - \sum\limits_{i=2}^{n}\kappa_{n-i+1}f_i} & (k = 1, \ \cdots, \ n-1), \\ A_{1n} = \dfrac{1}{\kappa_n - \sum\limits_{i=2}^{n}\kappa_{n-i+1}f_i} & \\ A_{ik} = f_{ik} - A_{1k}f_i & (i = 2, \ \cdots, \ n; \ k = 1, \ \cdots, \ n-1), \\ A_{in} = -A_{1n}f_i & \end{cases}$$

$$f_{ik} = \begin{cases} \dfrac{(-1)^{k+i-1}\Delta(k+1, i-1)}{\kappa_0^{i-k}} & (1 \leq k < i \leq n), \\ 0 & (i \leq k), \end{cases}$$

$$\Delta(i,j) = \begin{cases} \det T_H\left(\begin{array}{cccc}\kappa_i, & \kappa_{i+1}, & \cdots, & \kappa_j \\ \kappa_i, & \kappa_{i+1}, & \cdots, & \kappa_j\end{array}\right) & (1 \leq i \leq j \leq n), \\ 1 & (i > j), \end{cases}$$

$$f_i = \sum_{j=1}^{i-1}\frac{(-1)^{j+i-1}\kappa_j}{\kappa_0^{i-j}}\Delta(j+1, i-1).$$

Let n be a positive integer. Then the determinant of an $n \times n$ Toeplitz-Hessenberg matrix

$$\det T_H\left(\begin{array}{cccc}\kappa_i, & \kappa_{i+1}, & \cdots, & \kappa_j \\ \kappa_i, & \kappa_{i+1}, & \cdots, & \kappa_j\end{array}\right) = \sum_{\hat{t}_1+2\hat{t}_2+\cdots+n\hat{t}_n=n}\frac{(\hat{t}_1+\cdots+\hat{t}_n)!}{\hat{t}_1!\cdots\hat{t}_n!}(-\kappa_i)^{n-\hat{t}_1-\cdots-\hat{t}_n}\kappa_{i+1}^{\hat{t}_1}\kappa_{i+2}^{\hat{t}_2}\cdots\kappa_j^{\hat{t}_n},$$

where $\frac{(\hat{t}_1+\cdots+\hat{t}_n)!}{\hat{t}_1!\cdots\hat{t}_n!}$ *is the multinomial coefficient;* $n=\hat{t}_1+2\hat{t}_2+\cdots+n\hat{t}_n$ *is a partition of the positive integer* n *where each positive integer* i *appears* $\hat{t}_i$ *times.*

Lemma 1.2. *(see, e.g.[18] p.19) Let* D *and* $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be invertible matrices, we have*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}=\begin{pmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1}+D^{-1} \end{pmatrix},$$

where the Schur complement of D*, i.e.* $A-BD^{-1}C$*, is also invertible.*

2 Determinant and Inverse of the Skew Peoeplitz Matrix

Let $\mathbf{T}_{R,n}$ be an invertible skew Peoeplitz matrix. In this section, we give the determinant and inverse of the matrix $\mathbf{T}_{R,n}$.

Theorem 2.1. *Let* $\mathbf{T}_{R,n}$ *be an invertible skew Peoeplitz matrix the form of (1.1), we get*

$$\begin{aligned} &\det \mathbf{T}_{R,1}=3, \quad \det \mathbf{T}_{R,2}=9, \quad \det \mathbf{T}_{R,3}=39, \\ &\det \mathbf{T}_{R,4}=250, \quad \det \mathbf{T}_{R,5}=1548, \quad \det \mathbf{T}_{R,6}=11074, \end{aligned}$$

and

$$\begin{aligned} \det \mathbf{T}_{R,n} =&3(-1)^{n-2}\Bigg\{\Bigg(\sum_{k=1}^{n-4}(\frac{R_{n-1}R_k}{3}-R_{n-k-1})\ell_{n-k-1}+\frac{R_{n-1}}{3}(R_{n-1}+R_n)+1\Bigg) \\ &\cdot\Big(R_{n-3}\varpi_2-R_{n-4}\varpi_1+6\varpi_4\Big) \\ &-\Bigg(\sum_{k=1}^{n-4}(\frac{R_{n-2}R_k}{3}-R_{n-k-2})\ell_{n-k-1}+\frac{R_{n-2}}{3}(R_{n-1}+R_n)+3\Bigg) \\ &\cdot\Big(R_{n-2}\varpi_2-R_{n-4}\varpi_3+6\varpi_6\Big) \\ &+\Bigg(\sum_{k=1}^{n-4}(\frac{R_{n-3}R_k}{3}-R_{n-k-3})\ell_{n-k-1}+\frac{R_{n-3}}{3}(R_{n-1}+R_n)+5\Bigg) \\ &\cdot\Big(R_{n-2}\varpi_1-R_{n-3}\varpi_3+6\varpi_5\Big)\Bigg\} \ (n>6), \end{aligned} \tag{2.1}$$

where

$$\ell_0=\ell_1=\ell_2=1,\ \ell_i=-\frac{1}{6}(5\ell_{i-1}+2\ell_{i-2}+\sum_{k=0}^{i-3}\hbar_k\ell_{i-3-k}),\ (3\le i\le n-2), \tag{2.2}$$

and

$$\varpi_1 = \left(p_{n-2} - 2\sum_{i=1}^{n-4} a_{i1}p_{n-i-2} - \sum_{k=0}^{n-6} \hbar_k \sum_{i+1}^{n-4} a_{i,k+2}p_{n-i-2}\right)$$
$$\cdot \left(\sum_{\hat{t}_1+2\hat{t}_2+\cdots+n\hat{t}_n=n} \frac{(\hat{t}_1+\cdots+\hat{t}_n)!}{\hat{t}_1!\cdots\hat{t}_n!}(-6)^{n-\hat{t}_1-\cdots-\hat{t}_n}5^{\hat{t}_1}2^{\hat{t}_2}\cdots\hbar_{n-7}^{\hat{t}_n}\right), \quad (2.3)$$

$$\varpi_2 = \left(q_{n-2} - 2\sum_{i=1}^{n-4} a_{i1}q_{n-i-2} - \sum_{k=0}^{n-6} \hbar_k \sum_{i+1}^{n-4} a_{i,k+2}q_{n-i-2}\right)$$
$$\cdot \left(\sum_{\hat{t}_1+2\hat{t}_2+\cdots+n\hat{t}_n=n} \frac{(\hat{t}_1+\cdots+\hat{t}_n)!}{\hat{t}_1!\cdots\hat{t}_n!}(-6)^{n-\hat{t}_1-\cdots-\hat{t}_n}5^{\hat{t}_1}2^{\hat{t}_2}\cdots\hbar_{n-7}^{\hat{t}_n}\right), \quad (2.4)$$

$$\varpi_3 = \left(o_{n-2} - 2\sum_{i=1}^{n-4} a_{i1}o_{n-i-2} - \sum_{k=0}^{n-6} \hbar_k \sum_{i+1}^{n-4} a_{i,k+2}o_{n-i-2}\right)$$
$$\cdot \left(\sum_{\hat{t}_1+2\hat{t}_2+\cdots+n\hat{t}_n=n} \frac{(\hat{t}_1+\cdots+\hat{t}_n)!}{\hat{t}_1!\cdots\hat{t}_n!}(-6)^{n-\hat{t}_1-\cdots-\hat{t}_n}5^{\hat{t}_1}2^{\hat{t}_2}\cdots\hbar_{n-7}^{\hat{t}_n}\right), \quad (2.5)$$

$$\varpi_4 = p_{n-2}q_{n-3} + (p_{n-3} - q_{n-3})\left(2\sum_{i=1}^{n-5} m_i q_{n-i-3} + \sum_{k=0}^{n-7}\hbar_k \sum_{i=1}^{n-6-k} m_i q_{n-i-4-k} - q_{n-2} - 1\right)$$
$$+ \left(2\sum_{i=1}^{n-5} m_i(p_{n-i-3} - q_{n-i-3})\right) + \left(\sum_{k=0}^{n-7}\hbar_k \sum_{i=1}^{n-6-k} m_i(p_{n-i-4-k} - q_{n-i-4-k}) + q_{n-2} - p_{n-2}\right)$$
$$\cdot \left(5\sum_{i=1}^{n-5} m_i p_{n-i-3} + 2\sum_{i=1}^{n-6} m_i p_{n-i-4} + \sum_{k=0}^{n-8}\hbar_k \sum_{i=1}^{n-7-k} p_{n-i-5-k}\right), \quad (2.6)$$

$$\varpi_5 = o_{n-2}p_{n-3} + (o_{n-3} - p_{n-3})\left(2\sum_{i=1}^{n-5} m_i q_{n-i-3} + \sum_{k=0}^{n-7}\hbar_k \sum_{i=1}^{n-6-k} m_i q_{n-i-4-k} - q_{n-2} - 1\right)$$
$$+ \left(2\sum_{i=1}^{n-5} m_i(o_{n-i-3} - p_{n-i-3})\right) + \left(\sum_{k=0}^{n-7}\hbar_k \sum_{i=1}^{n-6-k} m_i(o_{n-i-4-k} - q_{n-i-4-k}) + p_{n-2} - o_{n-2}\right)$$
$$\cdot \left(5\sum_{i=1}^{n-5} m_i o_{n-i-3} + 2\sum_{i=1}^{n-6} m_i o_{n-i-4} + \sum_{k=0}^{n-8}\hbar_k \sum_{i=1}^{n-7-k} o_{n-i-5-k}\right), \quad (2.7)$$

$$\varpi_6 = o_{n-2}q_{n-3} + (o_{n-3} - q_{n-3})\Bigg(2\sum_{i=1}^{n-5} m_i q_{n-i-3} + \sum_{k=0}^{n-7} \hbar_k \sum_{i=1}^{n-6-k} m_i o_{n-i-4-k} - q_{n-2} - 1\Bigg)$$
$$+ \Bigg(2\sum_{i=1}^{n-5} m_i(o_{n-i-3} - q_{n-i-3})\Bigg) + \Bigg(\sum_{k=0}^{n-7} \hbar_k \sum_{i=1}^{n-6-k} m_i(o_{n-i-4-k} - q_{n-i-4-k}) + q_{n-2} - o_{n-2}\Bigg)$$
$$\cdot \Bigg(5\sum_{i=1}^{n-5} m_i o_{n-i-3} + 2\sum_{i=1}^{n-6} m_i o_{n-i-4} + \sum_{k=0}^{n-8} \hbar_k \sum_{i=1}^{n-7-k} o_{n-i-5-k}\Bigg), \tag{2.8}$$

with

$$\begin{aligned}
&o_i = \frac{R_{n-1}R_i}{3} - R_{n-i-1},\ (1 \le i \le n-2),\ o_{n-1} = \frac{R_{n-1}R_i}{3} + 3,\\
&p_i = \frac{R_{n-2}R_i}{3} - R_{n-i-2},\ (1 \le i \le n-3),\ p_{n-2} = \frac{R_{n-2}^2}{3} + 3,\\
&p_{n-1} = \frac{R_{n-2}R_{n-1}}{3},\\
&q_i = \frac{R_{n-3}R_i}{3} - R_{n-i-3},\ (1 \le i \le n-4),\ q_{n-3} = \frac{R_{n-3}^2}{3} + 3,\\
&q_{n-2} = \frac{R_{n-3}R_{n-2}}{3},\ q_{n-1} = \frac{R_{n-3}R_{n-1}}{3} + 2,\\
&\hbar_i = R_{i+4},\ (0 \le i \le n-5).
\end{aligned}$$

Proof. For $n \le 6$, it is easy to check that

$$\begin{aligned}
&\det \mathbf{T}_{R,1} = 3, && \det \mathbf{T}_{R,2} = 9, && \det \mathbf{T}_{R,3} = 39,\\
&\det \mathbf{T}_{R,4} = 250, && \det \mathbf{T}_{R,5} = 1548, && \det \mathbf{T}_{R,6} = 11074.
\end{aligned}$$

We can introduce the following two transformation matrices when $n > 6$,

$$\mathcal{S}_1 = \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\
\frac{R_{n-1}}{3} & \vdots & & & & & \iddots & 1\\
\frac{R_{n-2}}{3} & \vdots & & & & \iddots & 1 & 0\\
\frac{R_{n-3}}{3} & \vdots & & & \iddots & 1 & 0 & 0\\
0 & \vdots & & \iddots & 1 & 1 & 0 & -1\\
\vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & 0\\
\vdots & 0 & 1 & 1 & 0 & -1 & \iddots & \vdots\\
0 & 1 & 1 & 0 & -1 & 0 & \cdots & 0
\end{pmatrix}_{n\times n}, \tag{2.9}$$

and

$$\mathcal{B}_1 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ell_{n-2} & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ \vdots & \ell_{n-3} & \vdots & & & \iddots & 1 & 0 \\ \vdots & \ell_{n-4} & \vdots & & \iddots & 1 & \iddots & \vdots \\ \vdots & \vdots & \vdots & \iddots & \iddots & \iddots & & \vdots \\ \vdots & \ell_2 & 0 & \iddots & \iddots & & & \vdots \\ \vdots & \ell_1 & 1 & \iddots & & & & \vdots \\ 0 & \ell_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n\times n}, \tag{2.10}$$

where

$$\ell_0 = \ell_1 = \ell_2 = 1,\ \ell_i = -\frac{1}{6}(5\ell_{i-1} + 2\ell_{i-2} + \sum_{k=0}^{i-3} \hbar_k \ell_{i-3-k}),\ (3 \leqslant i \leqslant n-2).$$

By using (2.9), (2.10) and the recurrence relations of the Perrin sequences, the matrix $\mathbf{T}_{R,n}$ is changed into the following form,

$$\mathcal{S}_1\mathbf{T}_{R,n}\mathcal{B}_1 = \begin{pmatrix} 3 & Q_0 & R_{n-2} & R_{n-3} & R_{n-4} & R_{n-5} & \cdots & R_4 & R_3 & R_2 & R_1 \\ 0 & Q_1 & o_{n-2} & o_{n-3} & o_{n-4} & o_{n-5} & \cdots & o_4 & o_3 & o_2 & o_1 \\ \vdots & Q_2 & p_{n-2} & p_{n-3} & p_{n-4} & p_{n-5} & \cdots & p_4 & p_3 & p_2 & p_1 \\ \vdots & Q_3 & q_{n-2} & q_{n-3} & q_{n-4} & q_{n-5} & \cdots & q_4 & q_3 & q_2 & q_1 \\ \vdots & 0 & 2 & 5 & 6 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \hbar_0 & 2 & 5 & 6 & \ddots & & & & \vdots \\ \vdots & \vdots & \hbar_1 & \hbar_0 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \hbar_{n-7} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \hbar_{n-6} & \hbar_{n-7} & \cdots & \cdots & \hbar_1 & \hbar_0 & 2 & 5 & 6 \end{pmatrix}_{n\times n}, \tag{2.11}$$

where

$$
\begin{aligned}
&Q_0=\sum_{k=1}^{n-1} R_k\ell_{n-k-1},\ Q_1=\sum_{k=1}^{n-1} o_k\ell_{n-k-1},\ Q_2=\sum_{k=1}^{n-1} p_k\ell_{n-k-1},\ Q_3=\sum_{k=1}^{n-1} q_k\ell_{n-k-1},\\
&o_i=\frac{R_{n-1}R_i}{3}-R_{n-i-1},\ (1\le i\le n-2),\ o_{n-1}=\frac{R_{n-1}R_i}{3}+3,\\
&p_i=\frac{R_{n-2}R_i}{3}-R_{n-i-2},\ (1\le i\le n-3),\ p_{n-2}=\frac{R_{n-2}^2}{3}+3,\\
&p_{n-1}=\frac{R_{n-2}R_{n-1}}{3},\\
&q_i=\frac{R_{n-3}R_i}{3}-R_{n-i-3},\ (1\le i\le n-4),\ q_{n-3}=\frac{R_{n-3}^2}{3}+3,\\
&q_{n-2}=\frac{R_{n-3}R_{n-2}}{3},\ q_{n-1}=\frac{R_{n-3}R_{n-1}}{3}+2,\\
&\hbar_i=R_{i+4},\ (0\le i\le n-5).
\end{aligned}
$$

Do the Laplace expansion of matrix (2.11), we can get that

$$
\begin{aligned}
\det(\mathcal{S}_1\mathbf{T}_{R,n}\mathcal{B}_1)=3(-1)^{n-2}\Big\{&Q_1[-p_1\varpi_2+q_1\varpi_1+6\varpi_4]\\
&-Q_2[-o_1\varpi_2+q_1\varpi_3+6\varpi_6]+Q_3[-o_1\varpi_1+p_1\varpi_3+6\varpi_5]\Big\},
\end{aligned}
\tag{2.12}
$$

where

$$
\begin{aligned}
\varpi_1&=\det\Phi_{n-3}([p_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-6}),\\
\varpi_2&=\det\Phi_{n-3}([q_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-6}),\\
\varpi_3&=\det\Phi_{n-3}([o_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-6}),\\
\varpi_4&=\det\Phi_{n-3}([p_k]_{k=2}^{n-2},\ [q_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-7}),\\
\varpi_5&=\det\Phi_{n-3}([o_k]_{k=2}^{n-2},\ [p_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-7}),\\
\varpi_6&=\det\Phi_{n-3}([o_k]_{k=2}^{n-2},\ [q_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-7}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\Phi_{n-3}([p_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-6})\\
&=\begin{pmatrix}
p_{n-2} & p_{n-3} & p_{n-4} & p_{n-5} & p_{n-6} & \cdots & p_3 & p_2\\
2 & 5 & 6 & 0 & \cdots & \cdots & \cdots & 0\\
\hbar_0 & 2 & 5 & 6 & \ddots & & & \vdots\\
\hbar_1 & \hbar_0 & 2 & 5 & 6 & \ddots & & \vdots\\
\hbar_2 & \hbar_1 & \hbar_0 & 2 & 5 & 6 & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\
\vdots & & \ddots & \hbar_1 & \hbar_0 & 2 & 5 & 6\\
\hbar_{n-6} & \cdots & \cdots & \hbar_2 & \hbar_1 & \hbar_0 & 2 & 5
\end{pmatrix}_{(n-3)\times(n-3)},
\end{aligned}
\tag{2.13}
$$

$$\Phi_{n-3}([q_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-6})$$
$$= \begin{pmatrix} q_{n-2} & q_{n-3} & q_{n-4} & q_{n-5} & q_{n-6} & \cdots & q_3 & q_2 \\ 2 & 5 & 6 & 0 & \cdots & \cdots & \cdots & 0 \\ \hbar_0 & 2 & 5 & 6 & \ddots & & & \vdots \\ \hbar_1 & \hbar_0 & 2 & 5 & 6 & \ddots & & \vdots \\ \hbar_2 & \hbar_1 & \hbar_0 & 2 & 5 & 6 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \hbar_1 & \hbar_0 & 2 & 5 & 6 \\ \hbar_{n-6} & \cdots & \cdots & \hbar_2 & \hbar_1 & \hbar_0 & 2 & 5 \end{pmatrix}_{(n-3)\times(n-3)}, \tag{2.14}$$

$$\Phi_{n-3}([o_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-6})$$
$$= \begin{pmatrix} o_{n-2} & o_{n-3} & o_{n-4} & o_{n-5} & o_{n-6} & \cdots & o_3 & o_2 \\ 2 & 5 & 6 & 0 & \cdots & \cdots & \cdots & 0 \\ \hbar_0 & 2 & 5 & 6 & \ddots & & & \vdots \\ \hbar_1 & \hbar_0 & 2 & 5 & 6 & \ddots & & \vdots \\ \hbar_2 & \hbar_1 & \hbar_0 & 2 & 5 & 6 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \hbar_1 & \hbar_0 & 2 & 5 & 6 \\ \hbar_{n-6} & \cdots & \cdots & \hbar_2 & \hbar_1 & \hbar_0 & 2 & 5 \end{pmatrix}_{(n-3)\times(n-3)}, \tag{2.15}$$

$$\Phi_{n-3}([p_k]_{k=2}^{n-2},\ [q_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-7})$$
$$= \begin{pmatrix} p_{n-2} & p_{n-3} & p_{n-4} & p_{n-5} & p_{n-6} & \cdots & p_3 & p_2 \\ q_{n-2} & q_{n-3} & q_{n-4} & q_{n-5} & q_{n-6} & \cdots & q_3 & q_2 \\ 2 & 5 & 6 & 0 & \cdots & \cdots & \cdots & 0 \\ \hbar_0 & 2 & 5 & 6 & \ddots & & & \vdots \\ \hbar_1 & \hbar_0 & 2 & 5 & 6 & \ddots & & \vdots \\ \hbar_2 & \hbar_1 & \hbar_0 & 2 & 5 & 6 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \hbar_{n-7} & \cdots & \cdots & \hbar_1 & \hbar_0 & 2 & 5 & 6 \end{pmatrix}_{(n-3)\times(n-3)}, \tag{2.16}$$

$$\Phi_{n-3}([o_k]_{k=2}^{n-2},\ [p_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-7})$$
$$= \begin{pmatrix} o_{n-2} & o_{n-3} & o_{n-4} & o_{n-5} & o_{n-6} & \cdots & o_3 & o_2 \\ p_{n-2} & p_{n-3} & p_{n-4} & p_{n-5} & p_{n-6} & \cdots & p_3 & p_2 \\ 2 & 5 & 6 & 0 & \cdots & \cdots & \cdots & 0 \\ \hbar_0 & 2 & 5 & 6 & \ddots & & & \vdots \\ \hbar_1 & \hbar_0 & 2 & 5 & 6 & \ddots & & \vdots \\ \hbar_2 & \hbar_1 & \hbar_0 & 2 & 5 & 6 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \hbar_{n-7} & \cdots & \cdots & \hbar_1 & \hbar_0 & 2 & 5 & 6 \end{pmatrix}_{(n-3)\times(n-3)}, \tag{2.17}$$

$$\Phi_{n-3}([o_k]_{k=2}^{n-2},\ [q_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-7})$$
$$= \begin{pmatrix} o_{n-2} & o_{n-3} & o_{n-4} & o_{n-5} & o_{n-6} & \cdots & o_3 & o_2 \\ q_{n-2} & q_{n-3} & q_{n-4} & q_{n-5} & q_{n-6} & \cdots & q_3 & q_2 \\ 2 & 5 & 6 & 0 & \cdots & \cdots & \cdots & 0 \\ \hbar_0 & 2 & 5 & 6 & \ddots & & & \vdots \\ \hbar_1 & \hbar_0 & 2 & 5 & 6 & \ddots & & \vdots \\ \hbar_2 & \hbar_1 & \hbar_0 & 2 & 5 & 6 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \hbar_{n-7} & \cdots & \cdots & \hbar_1 & \hbar_0 & 2 & 5 & 6 \end{pmatrix}_{(n-3)\times(n-3)}. \tag{2.18}$$

According to Lemma 1.1 and Lemma 1.2, computing the determinant of the matrices on both sides of the equation (2.13)-(2.18), we get that

$$\begin{aligned} &\det \Phi_{n-3}([p_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-6}) \\ &\quad = \left(p_{n-2} - 2\sum_{i=1}^{n-4} a_{i1} p_{n-i-2} - \sum_{k=0}^{n-6} \hbar_k \sum_{i+1}^{n-4} a_{i,k+2} p_{n-i-2}\right) \\ &\qquad \cdot \left(\sum_{\hat{t}_1+2\hat{t}_2+\cdots+n\hat{t}_n=n} \frac{(\hat{t}_1+\cdots+\hat{t}_n)!}{\hat{t}_1!\cdots\hat{t}_n!}(-6)^{n-\hat{t}_1-\cdots-\hat{t}_n} 5^{\hat{t}_1} 2^{\hat{t}_2}\cdots \hbar_{n-7}^{\hat{t}_n}\right), \end{aligned} \tag{2.19}$$

$$\begin{aligned} &\det \Phi_{n-3}([q_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-6}) \\ &\quad = \left(q_{n-2} - 2\sum_{i=1}^{n-4} a_{i1} q_{n-i-2} - \sum_{k=0}^{n-6} \hbar_k \sum_{i+1}^{n-4} a_{i,k+2} q_{n-i-2}\right) \\ &\qquad \cdot \left(\sum_{\hat{t}_1+2\hat{t}_2+\cdots+n\hat{t}_n=n} \frac{(\hat{t}_1+\cdots+\hat{t}_n)!}{\hat{t}_1!\cdots\hat{t}_n!}(-6)^{n-\hat{t}_1-\cdots-\hat{t}_n} 5^{\hat{t}_1} 2^{\hat{t}_2}\cdots \hbar_{n-7}^{\hat{t}_n}\right), \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\det \Phi_{n-3}([o_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-6}) \\ &\quad = \left(o_{n-2} - 2\sum_{i=1}^{n-4} a_{i1} o_{n-i-2} - \sum_{k=0}^{n-6} \hbar_k \sum_{i+1}^{n-4} a_{i,k+2} o_{n-i-2}\right) \\ &\qquad \cdot \left(\sum_{\hat{t}_1+2\hat{t}_2+\cdots+n\hat{t}_n=n} \frac{(\hat{t}_1+\cdots+\hat{t}_n)!}{\hat{t}_1!\cdots\hat{t}_n!}(-6)^{n-\hat{t}_1-\cdots-\hat{t}_n} 5^{\hat{t}_1} 2^{\hat{t}_2}\cdots \hbar_{n-7}^{\hat{t}_n}\right), \end{aligned} \tag{2.21}$$

$$\begin{aligned} &\det \Phi_{n-3}([p_k]_{k=2}^{n-2},\ [q_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-7}) \\ &\quad = p_{n-2}q_{n-3} + (p_{n-3}-q_{n-3})\left(2\sum_{i=1}^{n-5} m_i q_{n-i-3} + \sum_{k=0}^{n-7}\hbar_k \sum_{i=1}^{n-6-k} m_i q_{n-i-4-k} - q_{n-2} - 1\right) \\ &\qquad + \left(2\sum_{i=1}^{n-5} m_i (p_{n-i-3} - q_{n-i-3})\right) + \left(\sum_{k=0}^{n-7}\hbar_k \sum_{i=1}^{n-6-k} m_i (p_{n-i-4-k} - q_{n-i-4-k}) + q_{n-2} - p_{n-2}\right) \\ &\qquad \cdot \left(5\sum_{i=1}^{n-5} m_i p_{n-i-3} + 2\sum_{i=1}^{n-6} m_i p_{n-i-4} + \sum_{k=0}^{n-8}\hbar_k \sum_{i=1}^{n-7-k} p_{n-i-5-k}\right), \end{aligned} \tag{2.22}$$

$$\begin{aligned}
&\det \Phi_{n-3}([o_k]_{k=2}^{n-2},\ [p_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-7}) \\
&\quad = o_{n-2}p_{n-3} + (o_{n-3} - p_{n-3})\Big(2\sum_{i=1}^{n-5} m_i q_{n-i-3} + \sum_{k=0}^{n-7}\hbar_k \sum_{i=1}^{n-6-k} m_i q_{n-i-4-k} - q_{n-2} - 1\Big) \\
&\qquad + \Big(2\sum_{i=1}^{n-5} m_i(o_{n-i-3} - p_{n-i-3})\Big) + \Big(\sum_{k=0}^{n-7}\hbar_k \sum_{i=1}^{n-6-k} m_i(o_{n-i-4-k} - q_{n-i-4-k}) + p_{n-2} - o_{n-2}\Big) \\
&\qquad \cdot \Big(5\sum_{i=1}^{n-5} m_i o_{n-i-3} + 2\sum_{i=1}^{n-6} m_i o_{n-i-4} + \sum_{k=0}^{n-8}\hbar_k \sum_{i=1}^{n-7-k} o_{n-i-5-k}\Big), \qquad (2.23)
\end{aligned}$$

$$\begin{aligned}
&\det \Phi_{n-3}([o_k]_{k=2}^{n-2},\ [q_k]_{k=2}^{n-2},\ 2,\ 5,\ 6,\ \hbar_0,\ \cdots,\ \hbar_{n-7}) \\
&\quad = o_{n-2}q_{n-3} + (o_{n-3} - q_{n-3})\Big(2\sum_{i=1}^{n-5} m_i q_{n-i-3} + \sum_{k=0}^{n-7}\hbar_k \sum_{i=1}^{n-6-k} m_i o_{n-i-4-k} - q_{n-2} - 1\Big) \\
&\qquad + \Big(2\sum_{i=1}^{n-5} m_i(o_{n-i-3} - q_{n-i-3})\Big) + \Big(\sum_{k=0}^{n-7}\hbar_k \sum_{i=1}^{n-6-k} m_i(o_{n-i-4-k} - q_{n-i-4-k}) + q_{n-2} - o_{n-2}\Big) \\
&\qquad \cdot \Big(5\sum_{i=1}^{n-5} m_i o_{n-i-3} + 2\sum_{i=1}^{n-6} m_i o_{n-i-4} + \sum_{k=0}^{n-8}\hbar_k \sum_{i=1}^{n-7-k} o_{n-i-5-k}\Big). \qquad (2.24)
\end{aligned}$$

While

$$\det \mathcal{S}_1 = \det \mathcal{B}_1 = (-1)^{\frac{(n-1)(n-2)}{2}}. \qquad (2.25)$$

From (2.12) and (2.25), we can obtain $\det \mathbf{T}_{R,n}$ as (2.1), which completes the proof. □

Theorem 2.2. *Let $\mathbf{T}_{R,n}$ be an invertible skew Peoeplitz matrix and $n > 6$. We have*

$$\mathbf{T}_{R,n}^{-1} = \begin{pmatrix}
\gamma_{1,1} & \gamma_{1,2} & \gamma_{1,3} & \cdots & \gamma_{1,n-2} & \gamma_{1,n-1} & \gamma_{1,n} \\
\gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & \cdots & \gamma_{2,n-2} & \gamma_{2,n-1} & \gamma_{1,n-1} \\
\gamma_{3,1} & \gamma_{3,2} & \gamma_{3,3} & \cdots & \gamma_{3,n-2} & \gamma_{2,n-2} & \gamma_{1,n-2} \\
\vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\
\gamma_{n-2,1} & \gamma_{n-2,2} & \gamma_{n-2,3} & \cdots & \gamma_{3,3} & \gamma_{2,3} & \gamma_{1,3} \\
\gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-2,2} & \cdots & \gamma_{3,2} & \gamma_{2,2} & \gamma_{1,2} \\
\gamma_{n,1} & \gamma_{n-1,1} & \gamma_{n-2,1} & \cdots & \gamma_{3,1} & \gamma_{2,1} & \gamma_{1,1}
\end{pmatrix}_{n\times n}, \qquad (2.26)$$

we can observed that $\mathbf{T}_{R,n}^{-1}$ is a symmetric matrix along secondary diagonal, i.e. sub-symmetric matrix.

where

$$
\begin{aligned}
\gamma_{1,1} &= \frac{1}{3} - \frac{Q_0 R_{n-1}}{9Q_1} + e_{n-2}(y_{11}\varphi_1 + y_{12}\varphi_2) + e_{n-3}(y_{21}\varphi_1 + y_{22}\varphi_2) \\
&\quad + \sum_{i=1}^{n-4} e_i(\beta_{n-i-3,1}\varphi_1 + \beta_{n-i-3,2}\varphi_2), \\
\gamma_{1,2} &= e_{n-2}\alpha_{1,n-4} + e_{n-3}\alpha_{2,n-4} + \sum_{i=1}^{n-4} e_i\theta_{n-i-3,n-4}, \\
\gamma_{1,j} &= e_{n-2}(\alpha_{1,n-j-1} + \alpha_{1,n-j-2}) + e_{n-3}(\alpha_{2,n-j-1} + \alpha_{2,n-j-2}) \\
&\quad + \sum_{i=1}^{n-4} e_i(\theta_{n-i-3,n-j-1} + \theta_{n-i-3,n-j-2}), (3 \le j \le 4), \\
\gamma_{1,j} &= e_{n-2}(\alpha_{1,n-j-1} + \alpha_{1,n-j-2} - \alpha_{1,n-j+1}) + e_{n-3}(\alpha_{2,n-j-1} + \alpha_{2,n-j-2} - \alpha_{2,n-j+1}) \\
&\quad + \sum_{i=1}^{n-4} e_i(\theta_{n-i-3,n-j-1} + \theta_{n-i-3,n-j-2} - \theta_{n-i-3,n-j+1}), (5 \le j \le n-3), \\
\gamma_{1,n-2} &= e_{n-2}(y_{12} + \alpha_{11} - \alpha_{13}) + e_{n-3}(y_{22} + \alpha_{21} - \alpha_{23}) + \sum_{i=1}^{n-4}(\beta_{n-i-3,2} + \theta_{n-i-3,1} - \theta_{n-i-3,3}), \\
\gamma_{1,n-1} &= e_{n-2}(y_{11} - \frac{Q_3}{Q_2}y_{12} - \alpha_{12}) + e_{n-3}(y_{21} - \frac{Q_3}{Q_2}y_{22} - \alpha_{22}) \\
&\quad + \sum_{i=1}^{n-4} e_i(\beta_{n-i-3,1} - \frac{Q_3}{Q_2}\beta_{n-i-3,2} - \theta_{n-i-3,2}), \\
\gamma_{1n} &= -\frac{Q_0}{3Q_1} + e_{n-2}(-\frac{Q_2}{Q_1}y_{11} - \alpha_{11}) + e_{n-3}(-\frac{Q_2}{Q_1}y_{21} - \alpha_{21}) \\
&\quad + \sum_{i=1}^{n-4} e_i(-\frac{Q_2}{Q_1}\beta_{n-i-3,1} - \theta_{n-i-3,1}), \\
\gamma_{k,1} &= \ell_{n-k}\frac{R_{n-1}}{3Q_1} - \frac{o_{n-2}}{Q_1}\ell_{n-k}(y_{11}\varphi_1 + y_{12}\varphi_2) - \frac{o_{n-3}}{Q_1}\ell_{n-k}(y_{21}\varphi_1 + y_{22}\varphi_2) \\
&\quad - \sum_{i=1}^{n-4} \frac{o_i}{Q_1}\ell_{n-k}(\beta_{n-i-3,1}\varphi_1 + \beta_{n-i-3,2}\varphi_2) + \tau_{n-k+2,1},\ (2 \le k \le n-1), \\
\gamma_{k,2} &= -\frac{o_{n-2}}{Q_1}\ell_{n-k}\alpha_{1,n-4} - \frac{o_{n-3}}{Q_1}\ell_{n-k}\alpha_{2,n-4} + \sum_{i=1}^{n-4}(-\frac{o_i}{Q_1}\ell_{n-k}\theta_{n-i-3,n-4}) \\
&\quad + \tau_{n-k+2,2},\ (2 \le k \le n-1), \\
\gamma_{k,j} &= -\frac{o_{n-2}}{Q_1}\ell_{n-k}(\alpha_{1,n-j-1} + \alpha_{1,n-j-2}) - \frac{o_{n-3}}{Q_1}\ell_{n-k}(\alpha_{2,n-j-1} + \alpha_{2,n-j-2}) \\
&\quad - \sum_{i=1}^{n-4} \frac{o_i}{Q_1}\ell_{n-k}(\theta_{n-i-3,n-j-1} + \theta_{n-i-3,n-j-2}) + \tau_{n-k+2,j},\ (3 \le j \le 4; 5-j \le k \le n-j+1), \\
\gamma_{k,j} &= -\frac{o_{n-2}}{Q_1}\ell_{n-k}(\alpha_{1,n-j-1} + \alpha_{1,n-j-2} - \alpha_{1,n-j+1}) - \frac{o_{n-3}}{Q_1}\ell_{n-k}(\alpha_{2,n-j-1} + \alpha_{2,n-j-2} - \alpha_{2,n-j+1}) \\
&\quad - \sum_{i=1}^{n-4} \frac{o_i}{Q_1}\ell_{n-k}(\theta_{n-i-3,n-j-1} + \theta_{n-i-3,n-j-2} - \theta_{n-i-3,n-j+1}) + \tau_{n-k+2,j}, \\
&\quad (5 \le j \le n-3; 7-j \le k \le n-j+1),
\end{aligned}
$$

$$\gamma_{k,n-2} = -\frac{o_{n-2}}{Q_1}\ell_{n-k}(y_{12}+\alpha_{11}-\alpha_{13}) - \frac{o_{n-3}}{Q_1}\ell_{n-k}(y_{22}+\alpha_{21}-\alpha_{23})$$
$$-\sum_{i=1}^{n-4}\frac{o_i}{Q_1}\ell_{n-k}(\beta_{n-i-3,2}+\theta_{n-i-3,1}-\theta_{n-i-3,3})+\tau_{n-j+2,n-2},\ (2\le k\le 3),$$
$$\gamma_{2,n-1} = -\frac{o_{n-2}}{Q_1}\ell_{n-2}(y_{11}-\frac{Q_3}{Q_2}y_{12}-\alpha_{12}) - \frac{o_{n-3}}{Q_1}\ell_{n-2}(y_{21}-\frac{Q_3}{Q_2}y_{22}-\alpha_{22})$$
$$-\sum_{i=1}^{n-4}\frac{o_i}{Q_1}\ell_{n-2}(\beta_{n-i-3,1}-\frac{Q_3}{Q_2}\beta_{n-i-3,2}-\theta_{n-i-3,2})+\tau_{n-j+2,n-1},$$
$$\gamma_{n,1} = \frac{R_{n-1}}{3Q_1}\ell_0 - \frac{o_{n-2}}{Q_1}\ell_0(y_{11}\varphi_1+y_{12}\varphi_2) - \frac{o_{n-3}}{Q_1}\ell_0(y_{21}\varphi_1+y_{22}\varphi_2)$$
$$-\sum_{i=1}^{n-4}\frac{o_i}{Q_1}\ell_0(\beta_{n-i-3,1}\varphi_1+\theta_{n-i-3,1}-\theta_{n-i-3,2}\varphi_2),$$

with

$$c_i = -\frac{Q_2}{Q_1}o_i + p_i,\ d_i = -\frac{Q_3}{Q_2}p_i + q_i, (1\le i\le n-2),$$
$$e_i = \frac{1}{3}(\frac{Q_0 o_i}{Q_1} - R_i), (1\le i\le n-2),$$
$$g_i = \sum_{k=1}^{n-i-3} m_k c_{n-i-k-2},\ f_i = \sum_{k=1}^{n-i-3} m_k d_{n-i-k-2},\ (1\le i\le n-4),$$
$$a_i = 2m_i + \sum_{k=0}^{i-2}\hbar_k m_{i-k-1},\ a_1 = 2m_1,\ (2\le i\le n-4),$$
$$b_i = 5m_i + 2m_{i-1} + \sum_{k=0}^{i-3}\hbar_k m_{i-k-2},\ b_1 = 5m_1,\ b_2 = 5m_2+2m_1,\ (3\le i\le n-4),$$
$$\theta_{i,j} = \begin{cases} (a_i y_{11}+b_i y_{21})g_j + (a_i y_{12}+b_i y_{22})f_j + m_{i-j+1}, & (1\le j\le i\le n-4), \\ (a_i y_{11}+b_i y_{21})g_j + (a_i y_{12}+b_i y_{22})f_j, & (1< i< j), \end{cases}$$
$$\alpha_{1,i} = -y_{11}g_i - y_{12}f_i, \alpha_{2,i} = -y_{21}g_i - y_{22}f_i,$$
$$\beta_{i,1} = -y_{11}a_i - y_{21}b_i, \beta_{i,2} = -y_{12}a_i - y_{22}b_i,$$
$$\varphi_1 = -\frac{R_{n-1}Q_2}{3Q_1} + \frac{R_{n-2}}{3}, \varphi_2 = -\frac{R_{n-2}Q_3}{3Q_2} + \frac{R_{n-3}}{3},$$
$$\tau_{11} = \frac{1}{3}, \tau_{21} = \frac{R_{n-1}}{3Q_1}, \tau_{2n} = \frac{1}{Q_1},$$
$$\tau_{ij} = \begin{cases} 0, (1\le i\le 2;\ 2\le j\le n;\ except\ \tau_{2n}), \\ \alpha_{i-2,n-j-1}+\alpha_{i-2,n-j-2},\ (3\le i\le 4;\ 3\le j\le 4), \\ \theta_{i-4,n-j-1}+\theta_{i-4,n-j-2},\ (5\le i\le n;\ 3\le j\le 4), \\ \alpha_{i-2,n-j-1}+\alpha_{i-2,n-j-2}+\alpha_{i-2,n-j+1},\ (3\le i\le 4;\ 5\le j\le n-3), \\ \theta_{i-4,n-j-1}+\theta_{i-4,n-j-2}-\theta_{i-4,n-j+1},\ (5\le i\le n;\ 5\le j\le n-3), \end{cases}$$
$$\tau_{i1} = \begin{cases} y_{i-2,1}\varphi_1 + y_{i-2,2}\varphi_2,\ (3\le i\le 4), \\ \beta_{i-4,1}\varphi_1 + \beta_{i-2,2}\varphi_2,\ (5\le i\le n-4), \end{cases}$$
$$\tau_{i2} = \begin{cases} \alpha_{i-2,n-4},\ (3\le i\le 4), \\ \theta_{i-4,n-4},\ (5\le i\le n-4), \end{cases}$$

$$\tau_{i,n-2} = \begin{cases} y_{i-2,2} + \alpha_{i-2,1} - \alpha_{i-2,3}, \ (3 \le i \le 4), \\ \beta_{i-4,2} + \theta_{i-4,1} - \theta_{i-4,3}, \ (5 \le i \le n), \end{cases}$$

$$\tau_{i,n-1} = \begin{cases} y_{i-2,1} - \frac{Q_3}{Q_2} y_{i-2,2} + \alpha_{i-2,2}, \ (3 \le i \le 4), \\ \beta_{i-4,1} - \frac{Q_3}{Q_2}\beta_{i-4,2} - \theta_{i-4,2}, \ (5 \le i \le n), \end{cases}$$

$$\tau_{in} = \begin{cases} -\frac{Q_2}{Q_1} y_{i-2,1} - \alpha_{i-2,1}, \ (3 \le i \le 4), \\ -\frac{Q_2}{Q_1}\beta_{i-4,1} - \theta_{i-4,1}, \ (5 \le i \le n). \end{cases}$$

Proof. Introduce the following two transformation matrices when $n > 6$,

$$\mathcal{S}_2 = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \vdots \\ \vdots & -\frac{Q_2}{Q_1} & 1 & \ddots & & & & \vdots \\ \vdots & 0 & -\frac{Q_3}{Q_2} & 1 & \ddots & & & \vdots \\ \vdots & \vdots & 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}_{n\times n}, \tag{2.27}$$

$$\mathcal{B}_2 = \begin{pmatrix} 1 & -\frac{Q_0}{3} & e_{n-2} & e_{n-3} & \cdots & e_2 & e_1 \\ 0 & 1 & -\frac{o_{n-2}}{Q_1} & -\frac{o_{n-3}}{Q_1} & \cdots & -\frac{o_2}{Q_1} & -\frac{o_1}{Q_1} \\ \vdots & \ddots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & 1 & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}_{n\times n}, \tag{2.28}$$

where

$$e_i = \frac{1}{3}\left(\frac{Q_0 o_i}{Q_1} - R_i\right), \ (1 \le i \le n-2).$$

If we multiply (2.11) by (2.27) from the left and (2.28) from the right, the (2.11) are as in the proof of Theorem 2.1, we obtain

$$\mathcal{S}_2\mathcal{S}_1\mathbf{T}_{R,n}\mathcal{B}_1\mathcal{B}_2=\begin{pmatrix} 3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & Q_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & c_{n-2} & c_{n-3} & c_{n-4} & c_{n-5} & \cdots & c_4 & c_3 & c_2 & c_1 \\ \vdots & \vdots & d_{n-2} & d_{n-3} & d_{n-4} & d_{n-5} & \cdots & d_4 & d_3 & d_2 & d_1 \\ \vdots & \vdots & 2 & 5 & 6 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \hbar_0 & 2 & 5 & \ddots & \ddots & & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \hbar_{n-6} & \hbar_{n-7} & \cdots & \cdots & \cdots & \hbar_0 & 2 & 5 & 6 \end{pmatrix}_{n\times n}, \tag{2.29}$$

where

$$c_i=-\frac{Q_2}{Q_1}o_i+p_i,\ d_i=-\frac{Q_3}{Q_2}p_i+q_i,\ (1\leq i\leq n-2).$$

The last matrix admits a block partition of the form

$$\mathcal{S}_2\mathcal{S}_1\mathbf{T}_{R,n}\mathcal{B}_1\mathcal{B}_2=\Omega_1\oplus\Omega_2, \tag{2.30}$$

where $\Omega_1=\begin{pmatrix} 3 & 0 \\ 0 & Q_1 \end{pmatrix}_{2\times 2}$ is a diagonal matrix, and Ω_2 is a Toeplitz-like matrix,

$$\Omega_2=\begin{pmatrix} c_{n-2} & c_{n-3} & c_{n-4} & c_{n-5} & \cdots & c_4 & c_3 & c_2 & c_1 \\ d_{n-2} & d_{n-3} & d_{n-4} & d_{n-5} & \cdots & d_4 & d_3 & d_2 & d_1 \\ 2 & 5 & 6 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \hbar_0 & 2 & 5 & \ddots & \ddots & & & & \vdots \\ \hbar_1 & \hbar_0 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \hbar_{n-7} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \hbar_{n-6} & \hbar_{n-7} & \cdots & \cdots & \hbar_1 & \hbar_0 & 2 & 5 & 6 \end{pmatrix}_{(n-2)\times(n-2)}, \tag{2.31}$$

$\Omega_1\oplus\Omega_2$ is the direct sum of Ω_1 and Ω_2. From (2.30), then we obtain

$$\mathbf{T}_{R,n}^{-1}=\mathcal{B}(\Omega_1^{-1}\oplus\Omega_2^{-1})\mathcal{S}, \tag{2.32}$$

where

$$\mathcal{S}=\mathcal{S}_2\mathcal{S}_1=\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \frac{R_{n-1}}{3} & \vdots & & & & & & & \iddots & 1 \\ \varphi_1 & \vdots & & & & & & \iddots & 1 & -\frac{Q_2}{Q_1} \\ \varphi_2 & \vdots & & & & & \iddots & 1 & -\frac{Q_3}{Q_2} & 0 \\ 0 & \vdots & & & & \iddots & 1 & 1 & 0 & -1 \\ \vdots & \vdots & & & \iddots & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & \vdots & & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \iddots & & \vdots \\ \vdots & 0 & \iddots & \iddots & \iddots & \iddots & \iddots & & & \vdots \\ 0 & 1 & 1 & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}_{n\times n}, \tag{2.33}$$

$$\mathcal{B}=\mathcal{B}_1\mathcal{B}_2=\begin{pmatrix} 1 & -\frac{Q_0}{3} & e_{n-2} & e_{n-3} & \cdots & e_2 & e_1 \\ 0 & \ell_{n-2} & -\frac{o_{n-2}\ell_{n-2}}{Q_1} & -\frac{o_{n-3}\ell_{n-2}}{Q_1} & \cdots & -\frac{o_2\ell_{n-2}}{Q_1} & -\frac{o_1\ell_{n-2}}{Q_1}+1 \\ \vdots & \ell_{n-3} & -\frac{o_{n-2}\ell_{n-3}}{Q_1} & -\frac{o_{n-3}\ell_{n-3}}{Q_1} & \cdots & -\frac{o_2\ell_{n-3}}{Q_1}+1 & -\frac{o_1\ell_{n-3}}{Q_1} \\ \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \iddots & \cdots & \vdots & \vdots \\ \vdots & \ell_1 & -\frac{o_{n-2}\ell_1}{Q_1}+1 & -\frac{o_{n-3}\ell_1}{Q_1} & \cdots & -\frac{o_2\ell_1}{Q_1} & -\frac{o_1\ell_1}{Q_1} \\ 0 & \ell_0 & -\frac{o_{n-2}\ell_0}{Q_1} & -\frac{o_{n-3}\ell_0}{Q_1} & \cdots & -\frac{o_2\ell_0}{Q_1} & -\frac{o_1\ell_0}{Q_1} \end{pmatrix}_{n\times n}. \tag{2.34}$$

We can observe that the inverse matrix of Ω_1 is

$$\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \dfrac{1}{Q_1} \end{pmatrix}_{2\times 2}, \tag{2.35}$$

$$\Omega_2=\left(\begin{array}{c|c} \mathcal{A} & \alpha \\ \hline \beta & \mathcal{G} \end{array}\right)=\left(\begin{array}{cc|ccccccc} c_{n-2} & c_{n-3} & c_{n-4} & c_{n-5} & \cdots & c_4 & c_3 & c_2 & c_1 \\ d_{n-2} & d_{n-3} & d_{n-4} & d_{n-5} & \cdots & d_4 & d_3 & d_2 & d_1 \\ \hline 2 & 5 & 6 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \hbar_0 & 2 & 5 & \ddots & \ddots & & & & \vdots \\ \hbar_1 & \hbar_0 & 2 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \hbar_1 & \hbar_0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \hbar_1 & \hbar_0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \hbar_{n-6} & \hbar_{n-7} & \hbar_{n-8} & \cdots & \cdots & \hbar_0 & 2 & 5 & 6 \end{array}\right)_{(n-2)\times(n-2)}.$$

According to the Lemma (1.2), we have

$$\Omega_2^{-1} = \begin{pmatrix} \frac{1}{\mathcal{A}-\alpha\mathcal{G}^{-1}\beta} & -\frac{1}{\mathcal{A}-\alpha\mathcal{G}^{-1}\beta}\alpha\mathcal{G}^{-1} \\ -\mathcal{G}^{-1}\beta\frac{1}{\mathcal{A}-\alpha\mathcal{G}^{-1}\beta} & \mathcal{G}^{-1}\beta\frac{1}{\mathcal{A}-\alpha\mathcal{G}^{-1}\beta}\alpha\mathcal{G}^{-1}+\mathcal{G}^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} y_{11} & y_{12} & \alpha_{11} & \cdots & \cdots & \alpha_{1,n-4} \\ y_{21} & y_{22} & \alpha_{21} & \cdots & \cdots & \alpha_{2,n-4} \\ \beta_{11} & \beta_{12} & \theta_{11} & \cdots & \cdots & \theta_{1,n-4} \\ \beta_{21} & \beta_{22} & \theta_{21} & \cdots & \cdots & \theta_{2,n-4} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \beta_{n-4,1} & \beta_{n-4,2} & \theta_{n-4,1} & \cdots & \cdots & \theta_{n-4,n-4} \end{pmatrix}_{(n-2)\times(n-2)}, \tag{2.36}$$

where

$$\begin{aligned} &\alpha_{1i} = -y_{11}g_i - y_{12}f_i,\ \alpha_{2i} = -y_{21}g_i - y_{22}f_i, \\ &\beta_{i1} = -y_{11}a_i - y_{21}b_i,\ \alpha_{i2} = -y_{12}a_i - y_{22}b_i, \\ &x_{11} = c_{n-2} - 2g_1 - \sum_{i=0}^{n-6}\hbar_i g_{i+2},\ x_{12} = c_{n-3} - 5g_1 - 2g_2 - \sum_{i=0}^{n-7}\hbar_i g_{i+3}, \\ &x_{21} = d_{n-2} - 2f_1 - \sum_{i=0}^{n-6}\hbar_i f_{i+2},\ x_{22} = d_{n-3} - 5f_1 - 2f_2 - \sum_{i=0}^{n-7}\hbar_i f_{i+3}, \\ &w = x_{11} - \frac{x_{12}x_{21}}{x_{22}}, y_{11} = \frac{1}{w},\ y_{12} = -\frac{x_{12}}{wx_{22}},\ y_{21} = \frac{x_{21}}{wx_{22}},\ y_{22} = \frac{x_{12}x_{21}}{wx_{22}^2} + \frac{1}{x_{22}}, \\ &\theta_{i,j} = \begin{cases} (a_iy_{11} + b_iy_{21})g_j + (a_iy_{12} + b_iy_{22})f_j + m_{i-j+1}, & (1 \le j \le i \le n-4), \\ (a_iy_{11} + b_iy_{21})g_j + (a_iy_{12} + b_iy_{22})f_j, & (1 < i < j). \end{cases} \end{aligned}$$

Combineing (2.35) and (2.36), we obtain

$$\Omega_1^{-1} \oplus \Omega_2^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \frac{1}{Q_1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & y_{11} & y_{12} & \alpha_{11} & \cdots & \cdots & \alpha_{1,n-4} \\ \vdots & \vdots & y_{21} & y_{22} & \alpha_{21} & \cdots & \cdots & \alpha_{2,n-4} \\ \vdots & \vdots & \beta_{11} & \beta_{12} & \theta_{11} & \cdots & \cdots & \theta_{1,n-4} \\ \vdots & \vdots & \beta_{21} & \beta_{22} & \theta_{21} & \ddots & \ddots & \theta_{2,n-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \beta_{n-4,1} & \beta_{n-4,2} & \theta_{n-4,1} & \cdots & \cdots & \theta_{n-4,n-4} \end{pmatrix}_{n\times n}. \tag{2.37}$$

According to (2.32), (2.33), (2.34) and (2.37), we have

$$\mathbf{T}_{R,n}^{-1} = \mathcal{B}(\Omega_1^{-1} \oplus \Omega_2^{-1})\mathcal{S}$$
$$= \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} & \gamma_{1,3} & \cdots & \gamma_{1,n-2} & \gamma_{1,n-1} & \gamma_{1,n} \\ \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & \cdots & \gamma_{2,n-2} & \gamma_{2,n-1} & \gamma_{1,n-1} \\ \gamma_{3,1} & \gamma_{3,2} & \gamma_{3,3} & \cdots & \gamma_{3,n-2} & \gamma_{2,n-2} & \gamma_{1,n-2} \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ \gamma_{n-2,1} & \gamma_{n-2,2} & \gamma_{n-2,3} & \cdots & \gamma_{3,3} & \gamma_{2,3} & \gamma_{1,3} \\ \gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-2,2} & \cdots & \gamma_{3,2} & \gamma_{2,2} & \gamma_{1,2} \\ \gamma_{n,1} & \gamma_{n-1,1} & \gamma_{n-2,1} & \cdots & \gamma_{3,1} & \gamma_{2,1} & \gamma_{1,1} \end{pmatrix}_{n\times n}, \qquad (2.38)$$

where $\gamma_{i,j}$ $(1 \le i \le n;\ 1 \le j \le n-i+1)$, are the same as in Theorem 2.2, we can observed that $\mathbf{T}_{R,n}^{-1}$ is a sub-symmetric matrix. In this situation, we only need to work out $\frac{n(n+1)}{2}$ entries, hence it is easy to get the inverse of $\mathbf{T}_{R,n}$ by (2.38), which completes the proof. □

3 Determinant and Inverse of the Skew Peankel Matrix

Let $\mathbf{H}_{R,n}$ be an invertible skew Peankel matrix. In this section, we give the determinant and inverse of the matrix $\mathbf{H}_{R,n}$.

Theorem 3.1. *Let* $\mathbf{H}_{R,n}$ *be a skew Peankel matrix as the form of* (1.2)*. Then we have*

$$\det \mathbf{H}_{R,1} = 3, \quad \det \mathbf{H}_{R,2} = -9, \quad \det \mathbf{H}_{R,3} = -39,$$
$$\det \mathbf{H}_{R,4} = 250, \ \det \mathbf{H}_{R,5} = 1548, \ \det \mathbf{H}_{R,6} = -11074,$$

and

$$\begin{aligned} \det \mathbf{H}_{R,n} =& 3(-1)^{\frac{n^2+n-4}{2}} \Bigg\{ \Bigg(\sum_{k=1}^{n-4} (\frac{R_{n-1}R_k}{3} - R_{n-k-1})\ell_{n-k-1} + \frac{R_{n-1}}{3}(R_{n-1}+R_n) + 1 \Bigg) \\ & \cdot \Big(R_{n-3}\varpi_2 - R_{n-4}\varpi_1 + 6\varpi_4 \Big) \\ & - \Bigg(\sum_{k=1}^{n-4} (\frac{R_{n-2}R_k}{3} - R_{n-k-2})\ell_{n-k-1} + \frac{R_{n-2}}{3}(R_{n-1}+R_n) + 3 \Bigg) \\ & \cdot \Big(R_{n-2}\varpi_2 - R_{n-4}\varpi_3 + 6\varpi_6 \Big) \\ & + \Bigg(\sum_{k=1}^{n-4} (\frac{R_{n-3}R_k}{3} - R_{n-k-3})\ell_{n-k-1} + \frac{R_{n-3}}{3}(R_{n-1}+R_n) + 5 \Bigg) \\ & \cdot \Big(R_{n-2}\varpi_1 - R_{n-3}\varpi_3 + 6\varpi_5 \Big) \Bigg\} \ (n > 6), \end{aligned} \qquad (3.1)$$

where ℓ_i $(0 \le i \le n-2)$ *are the same as* (2.2) *and* ϖ_i $(1 \le i \le 6)$ *are the same as* (2.19)-(2.24) .

Proof. In the case $n > 6$, from (1.3), it follows that $\det \mathbf{H}_{R,n} = \det \mathbf{T}_{R,n} \det \hat{I}_n$. Then we can obtain (3.1) by using Theorem 2.1 and $\det \hat{I}_n = (-1)^{\frac{n(n-1)}{2}}$. □

Theorem 3.2. *Let $\mathbf{H}_{R,n}$ be an invertible skew Peankel matrix and $n > 6$. Then we have*

$$\mathbf{H}_{R,n}^{-1} = \begin{pmatrix} \gamma_{n,1} & \gamma_{n-1,1} & \gamma_{n-2,1} & \cdots & \gamma_{3,1} & \gamma_{2,1} & \gamma_{1,1} \\ \gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-2,2} & \cdots & \gamma_{3,2} & \gamma_{2,2} & \gamma_{1,2} \\ \gamma_{n-2,1} & \gamma_{n-2,2} & \gamma_{n-2,3} & \cdots & \gamma_{3,3} & \gamma_{2,3} & \gamma_{1,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \gamma_{3,1} & \gamma_{3,2} & \gamma_{3,3} & \cdots & \gamma_{3,n-2} & \gamma_{2,n-2} & \gamma_{1,n-2} \\ \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & \cdots & \gamma_{2,n-2} & \gamma_{2,n-1} & \gamma_{1,n-1} \\ \gamma_{1,1} & \gamma_{1,2} & \gamma_{1,3} & \cdots & \gamma_{1,n-2} & \gamma_{1,n-1} & \gamma_{1,n} \end{pmatrix}_{n\times n}, \tag{3.2}$$

where $\gamma_{i,j}$ $(1 \le i \le n;\ 1 \le j \le n-j+1)$, are the same as in Theorem 2.2, we can observed that $\mathbf{H}_{R,n}^{-1}$ is a symmetric matrix . In this situation, we only need to work out $\frac{n(n+1)}{2}$ entries, hence it is easy to get the inverse of $\mathbf{H}_{R,n}$ by (3.2).

Proof. We can obtain this conclusion by using (1.3) and Theorem 2.2. □

4 Conclusion

In this paper, by constructing the special transformation matrices we give the determinant and inverse of the skew Peoeplitz matrix and the skew Peankel matrix in section 2 and section 3.

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Competing Interests

The authors declare that no competing interests exist.

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