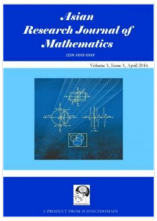

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Proof of Riemann Hypothesis

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Author's contribution

This work was carried out by author Robert Deloin alone. He managed the analysis, wrote the first draft of the manuscript and managed literature searches.

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Abstract

Riemann hypothesis is a conjecture that real part of every non-trivial zero of the Riemann zeta function is 1/2.
The main contribution of this paper is to achieve the proof of Riemann hypothesis. The key idea is to provide an Hamiltonian operator whose real eigenvalues correspond to the imaginary parts of the non-trivial zeros of Riemann zeta function and whose existence, according to Hilbert and Pólya, proves Riemann hypothesis.

Keywords: Euler; riemann; hilbert; polya; conjecture.

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1 Introduction

In his book [1] of 1748, Leonhard Euler (1707-1783) proved what is now named *the Euler product formula*. This product is the result of the following infinite sum.

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\mathbb{P}}(1-1/p^s)^{-1} \quad \text{for any integer variable } s>1$$

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where $\mathbb{P}$ is the infinite set of primes.

In his article [2] of 1859, Riemann (1826-1866) extended the Euler definition to the complex variable s of the zeta function.

$$\zeta(s) = \prod_{p\in\mathbb{P}}(1 - 1/p^s)^{-1} \quad \textit{for any complex variable } s \neq 1$$

It is known that the trivial zeros of the function are the infinite set

$$\{s_1\} = \{-2m\} \quad \text{for all integers } m > 0.$$

Riemann hypothesis can be seen as stating that

Probably, the infinite set of the non-trivial zeros $\{s_2\}$ of $\zeta(s)$ can be written as

$$\{s_2\} = \{\tfrac{1}{2} + it_n\} \quad \text{where } t_n \text{ is real.}$$

This conjecture is the first point of the eighth unresolved problem (among 23) that Hilbert listed in 1900 [3] as well as the second unresolved problem listed in 2000 by The Clay Mathematics Institute [4]. It has been very well described by Bombieri [5] and various approaches were listed in 2003 by Conrey [6] who says the following in his conclusion.

"A major difficulty in trying to construct a proof of RH through analysis is that the zeros of L-functions behave so much differently from zeros of many of the special functions we are used to seeing in mathematics and mathematical physics.

It is my belief that RH is a genuinely arithmetic question that likely will not succumb to methods of analysis."

That is why in the present paper, the many attempts of proof based on methods of analysis are consciously ignored and an approach is chosen, based on an operator point-of-view suggested by a discussion that occured in the 1910s between two mathematicians: Landau and Pólya. This discussion, whose track has been traced back by Andrew Odlyzko [7] in 1981-1982, gave the statement of the next section.

2 Preliminary Notes

2.1 The Hilbert-Pólya statement

Circa 1914 and independently from each other, Hilbert et Pólya[8] have orally stated that Riemann hypothesis would be proved if it could be shown that the imaginary parts t_n of the non-trivial zeros of the symmetrical xi function $\xi(s)$ derived from $\zeta(s)$, corresponded to the real eigenvalues of an unbounded self-adjoint Hamiltonian operator (here named $\hat{H}_\xi$) for which we could write:

$$\hat{H}_\xi \psi_k = E_k \psi_k \tag{2.1}$$

which is an equation of quantum physics where E_k stands for the kD-components in a kD-space of a physical energy, with $k = \infty$.

So, the first and unique purely mathematical clue that we have is that the operator $\hat{H}_\xi$ should be a square matrix of infinite dimension with real eigenvalues. This means that it could be written:

$$\hat{H}_\xi = (t_n) = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & t_{n-1} & 0 & 0 & 0 & \dots \\ \dots & 0 & t_n & 0 & 0 & \dots \\ \dots & 0 & 0 & t_{n+1} & 0 & \dots \\ \dots & 0 & 0 & 0 & t_{n+2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad \text{all } t_n \text{ being real}$$

3 Proof of Riemann Hypothesis

As Hilbert-Pólya statement will be used to prove Riemann Hypothesis, the complete proof will be established in three steps:

The first one proves that Hilbert-Pólya statement is indeed a conditional proof of Riemann Hypothesis.

The second one is a preparation for the third one.

The third one establishes the unconditional proof of Riemann Hypothesis.

3.1 Proof of Hilbert-Pólya statement

Proof. By definition, a complex number s is written:

$$s = x + iy \quad \text{where } x \text{ and } y \text{ are real and } i = \sqrt{-1}$$

By changing the conventional system of coordinates (x, y) of the complex plane into the new one $(x' = \frac{1}{2} - x, y' = y)$, these complex numbers can be written:

$$s = x' + iy' \text{ in the new system}$$
or:
$$s = (\tfrac{1}{2} - x) + iy \text{ when using the change of coordinates.}$$

Condition. We suppose that the $\hat{H}_\xi$ operator exists and that it contains the infinitely many real eigenvalues t_n coming from the non-trivial zeros s_2 of $\zeta(s)$.

Hypothesis. We then suppose that these non-trivial zeros can lie anywhere in the complex plane with the two exceptions that they cannot lie on the real axis x or x' (reserved for trivial zeros s_1), which gives:

$$y \neq 0 \text{ and } y' \neq 0$$

nor on the conventional critical line $x = \frac{1}{2}$ that becomes the new imaginary axis y', which gives:

$$x \neq \tfrac{1}{2} \text{ and } x' \neq 0$$

Then, *each non-trivial zero* s_2 *of* $\zeta(s)$ could be written:

$$s_2 = x'_2 + iy'_2 \quad \text{with } x'_2 \neq 0 \text{ and } y'_2 \neq 0$$
or, using the change of coordinates:
$$s_2 = (\tfrac{1}{2} - x_2) + iy_2 \quad \text{with } x_2 \neq \tfrac{1}{2} \text{ and } y_2 \neq 0$$

Now, using the fact that $-x_2 = i^2 x_2$, they can be written:

$$s_2 = (\tfrac{1}{2} + i^2 x_2) + iy_2 = \tfrac{1}{2} + i(y_2 + ix_2) \quad \text{with } x_2 \neq \tfrac{1}{2} \text{ and } y_2 \neq 0$$
or:
$$s_2 = \tfrac{1}{2} + it_2 \quad \text{with } t_2 = y_2 + ix_2,\ x_2 \neq \tfrac{1}{2} \text{ and } y_2 \neq 0$$

and we get the result, as $t_2 = y_2 + ix_2$ has to be real, that x_2 has to be zero. This is not a direct contradiction to the hypothesis but this result has been proven wrong 10^{13} times with the first 10^{13} non- trivial zeros s_2 [9] for which $x_2 = 1/2$. Each of these 10^{13} contradictions proves that our hypothesis is wrong and that Riemann hypothesis is true conditionally to the existence of the $\hat{H}_\xi$ operator, which is exactly Hilbert-Pólya statement. □

3.2 Preparing the unconditional proof of Riemann Hypothesis

As Riemann Hypothesis is now proven conditionally to the existence of the $\hat{H}_\xi$ operator, we have to prove that the $\hat{H}_\xi$ operator *does exist.*

To do this, we first notice that this operator is not a full link to $\zeta(s)$ as it refers only to its second set of non-trivial zeros $\{s_2\}$. We will therefore consider the new and larger operator $\hat{H}_\zeta$ built with the zeros of both sets $\{s_1\}$ and $\{s_2\}$ as eigenvalues, an operator that also contains the real values t_n (but not as eigenvalues):

$$\hat{H}_\zeta = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -6 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & -4 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & -2 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & \frac{1}{2}+it_1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & \frac{1}{2}+it_2 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}+it_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

As this new operator contains the real values t_n, it enables us, at any time, to rebuild the operator $\hat{H}_\xi$ of Hilbert and Pólya. To simplify the writing, we set:

$$(0) \text{ for all necessary zero values on and outside the diagonal,}$$
$$\begin{pmatrix} \dots & \dots & \dots & \dots \\ \dots & -6 & 0 & 0 \\ \dots & 0 & -4 & 0 \\ \dots & 0 & 0 & -2 \end{pmatrix} = (-2m)$$

and:

$$\begin{pmatrix} \frac{1}{2}+it_1 & 0 & 0 & \dots \\ 0 & \frac{1}{2}+it_2 & 0 & \dots \\ 0 & 0 & \frac{1}{2}+it_3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} = \left(\tfrac{1}{2}+it_n\right)$$

so that $\hat{H}_\zeta$ can be written:

$$\hat{H}_\zeta = \begin{pmatrix} (-2m) & (0) \\ (0) & \left(\frac{1}{2}+it_n\right) \end{pmatrix} = \begin{pmatrix} (-2m) & (0) \\ (0) & \left(\frac{1}{2}\right) \end{pmatrix} + i \begin{pmatrix} (0) & (0) \\ (0) & \hat{H}_\xi \end{pmatrix}$$

Now, the matrices $(-2m)$ and $\left(\frac{1}{2}+it_n\right)$ representing the sets of zeros $\{s_1\}$ and $\{s_2\}$ can symbolically be replaced by their parametric form:

$$-2m, \quad m > 0 \text{ being an integer parameter}$$
$$\tfrac{1}{2}+it_n, \quad t_n \text{ being a real parameter}$$

The sets $\{s_1\}$ and $\{s_2\}$ can then be considered as the two infinite sets of roots of the polynomial of complex variable s:

$$P(s) = (s - s_1)(s - s_2) = s^2 - (s_1 + s_2)s + s_1 s_2$$
$$P(s, m, t_n) = s^2 - (-2m + \tfrac{1}{2} + it_n)s - 2m(\tfrac{1}{2} + it_n)$$
$$P(s, m, t_n) = s^2 + (2m - (\tfrac{1}{2} + it_n))s - 2m(\tfrac{1}{2} + it_n)$$

which, using matrices, can be written either:

$$P(s, m, t_n) = \begin{pmatrix} s^2 & s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m - (\frac{1}{2} + it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2} + it_n) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{3.1}$$

or:

$$P(s, m, t_n) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m - (\frac{1}{2} + it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2} + it_n) \end{pmatrix} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} \tag{3.2}$$

By setting:

$$\hat{E}_k = \begin{pmatrix} s^2 & s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2m - (\frac{1}{2} + it_n)) & 0 \\ 0 & 0 & -2m(\frac{1}{2} + it_n) \end{pmatrix}$$
$$= \begin{pmatrix} s^2 & (2m - (\frac{1}{2} + it_n))s & -2m(\frac{1}{2} + it_n) \end{pmatrix}$$

and:

$$\psi_{E_k} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and, by multiplying its first two matrices, equation (3.1) gives:

$$P(s, m, t_n) = \begin{pmatrix} s^2 & (2m - (\frac{1}{2} + it_n))s & -2m(\frac{1}{2} + it_n) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \hat{E}_k \psi_{E_k} \tag{3.3}$$

where k is now reduced to $k = 3$ so that our initial problem is also reduced to our 3D space. As by multiplying the first two matrices of (3.2), we also have:

$$P(s, m, t_n) = \begin{pmatrix} 1 & (2m - (\frac{1}{2} + it_n)) & -2m(\frac{1}{2} + it_n) \end{pmatrix} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = \hat{H}_k \psi_{H_k} \tag{3.4}$$

when we set:

$$\hat{H}_k = \begin{pmatrix} 1 & (2m - (\frac{1}{2} + it_n)) & -2m(\frac{1}{2} + it_n) \end{pmatrix} \tag{3.5}$$

and:

$$\psi_{H_k} = \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix} = \hat{R} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \hat{R} \psi_{E_k}$$

where $\hat{R}$ is the 3-dimensional transformation matrix from the orthogonal system of coordinates ψ_{H_k} used to describe $\hat{H}_k$ to the orthogonal system ψ_{E_k} used to describe $\hat{E}_k$, and we have:

$$\hat{H}_k \psi_{H_k} = \hat{H}_k \hat{R} \psi_{E_k} = \hat{E}_k \psi_{E_k}$$

Then, setting $\hat{H} = \hat{H}_k \hat{R}$, we get:

$$\hat{H} \psi_{E_k} = \hat{E}_k \psi_{E_k} \tag{3.6}$$

which is almost identical to equation (2.1). Here, if the words *almost identical* are used, it is because $\hat{E}_k$ is an operator that may not be real as required by equation (2.1). That is why in the next section we look for the conditions that could make it real.

3.3 Unconditional proof of Riemann hypothesis

Proof. From equation (3.6) we have:

$$\hat{E}_k = \hat{H} = \hat{H}_k \hat{R}$$

and as from equation (3.5), $\hat{H}_k$ can also be written:

$$\hat{H}_k = \begin{pmatrix}1 & 1 & 1\end{pmatrix} \begin{pmatrix}1 & 0 & 0\\ 0 & (2m - (\frac{1}{2} + it_n)) & 0\\ 0 & 0 & -2m(\frac{1}{2} + it_n)\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\end{pmatrix} \hat{A}$$

when we set:

$$\hat{A} = \begin{pmatrix}1 & 0 & 0\\ 0 & (2m - (\frac{1}{2} + it_n)) & 0\\ 0 & 0 & -2m(\frac{1}{2} + it_n)\end{pmatrix} \tag{3.7}$$

we get from (3.2) and (3.1) that:

$$\begin{pmatrix}1 & 1 & 1\end{pmatrix} \hat{A} \begin{pmatrix}s^2\\ s\\ 1\end{pmatrix} = P(s, m, t_n) = \begin{pmatrix}s^2 & s & 1\end{pmatrix} \hat{A} \begin{pmatrix}1\\ 1\\ 1\end{pmatrix} \tag{3.8}$$

But, for any $s = x + iy$, we have:

$$P(s) = (s - s_1)(s - s_2) = s^2 - (s_1 + s_2)s + s_1 s_2$$

and, using parameters:

$$\begin{aligned}
P(s, m, t_n) &= s^2 + \left(2m - (\frac{1}{2} + it_n)\right) s - 2m \left(\frac{1}{2} + it_n\right)\\
&= (x + iy)^2 + \left(2m - (\frac{1}{2} + it_n)\right)(x + iy) - 2m \left(\frac{1}{2} + it_n\right)\\
&= \left(x^2 - y^2 + (2m - \frac{1}{2})x + yt_n\right) + i\left(-xt_n + y(2m - \frac{1}{2})\right) - m - 2mit_n\\
&= \left(x^2 - y^2 + (2m - \frac{1}{2})x + yt_n - m\right) + i\left(-xt_n + y(2x + 2m - \frac{1}{2}) - 2mt_n\right)
\end{aligned}$$

so that $P(s, m, t_n)$ will be real only when:

$$-xt_n + y(2x + 2m - \tfrac{1}{2}) - 2mt_n = 0$$

and so, only for the infinitely many curves in the complex plane such that:

$$y = t_n \frac{x + 2m}{2x + 2m - \frac{1}{2}} = t_n \frac{x + 2m}{(x + 2m) + (x - \frac{1}{2})}$$

which, for $x = \frac{1}{2}$, are all at $y = t_n$

and for $x = -2m$, are all at $y = 0$. Then, for all the points of all these (hyperbole-like) curves, we have that:

$$P(s, m, t_n)_{curves} = \left(x^2 - y^2 + (2m - \frac{1}{2})x + yt_n - m\right) = V(x, y) \tag{3.9}$$

is a real value and therefore the real mono-term matrix $(V(x, y))$ always verifies:

$$(V(x, y)) = (\overline{V(x, y)}) = (\overline{V(x, y)})^T \tag{3.10}$$

where $(\overline{V(x, y)})$ is the conjugate matrix of $(V(x, y))$ and $(\overline{V(x, y)})^T$ is the conjugate transpose of $(V(x, y))$. So, from (3.8), (3.9) and (3.10), we can write:

$$P(s,m,t_n)_{curves} = V(x,y) = \begin{pmatrix}1 & 1 & 1\end{pmatrix} \hat{A} \begin{pmatrix}s^2 \\ s \\ 1\end{pmatrix} = \left(\begin{pmatrix}s^2 & s & 1\end{pmatrix} \hat{A} \begin{pmatrix}1 \\ 1 \\ 1\end{pmatrix} \right)^T$$

which proves that the operator $\hat{A}$, also providing the real t_n's to $\hat{H}_\xi$, verifies the equation of the observables in quantum physics, which is generally written:

$$< \psi_1 \,|\hat{A}|\, \psi_2 > \; = \; \left(< \psi_2 \,|\hat{A}|\, \psi_1 >\right)^T$$

where $\hat{A}$ is the Hamiltonian operator associated to the physical quantity $A = V(x,y)$, $< x \mid$ and $\mid x >$ are the bra and ket operators (see[10]) on x, and ψ_1 and ψ_2 are the states of the real physical quantity A before and after the measuring of A.

As we have:

$$\hat{E}_k = \hat{H} = \hat{H}_k \hat{R} = \begin{pmatrix}1 & 1 & 1\end{pmatrix} \hat{A}\hat{R}$$

and as $\hat{A}$ can be associated with a real physical quantity $A = V(x,y)$, $\hat{E}_k = \hat{H}$ can also be associated with a real physical quantity E_k in equation (3.6), this last one becoming identical to (2.1).

As we can rebuild the Hamiltonian operator $\hat{H}_\xi$ linked to $\zeta(s)$ via the function $P(s,m,t_n)$ and the existing operator $\hat{H}$ or $\hat{E}_k$, this Hamiltonian operator $\hat{H}_\xi$ *does exist* and as we have proven earlier that Riemann hypothesis is true conditionally to the existence of the $\hat{H}_\xi$ operator, Riemann hypothesis is therefore unconditionally proven. □

4 Conclusion

It has to be noticed and highlighted that this proof does not use, as guessed by Conrey [6], any of the classical and internal analysis of $\zeta(s)$ with integrals, Fourier series, probabilities, etc. It uses only:

- complex numbers, in response to the complex variable s used by Riemann and to prove Hilbert-Pólya statement;

- parameters and polynomials, a side-step to replace the zeros of $\zeta(s)$ by parameters and to move the initial problem, already down-sized to matrices, to a simpler one with polynomials;

- operators and matrices, in response to equation (2.1) and to the eigenvalues cited in the Hilbert-Pólya statement and to express a polynomial as a product of matrices in a 3-dimensional space, and finally to find a self-adjoint hermitian operator that makes a link to a real physical observable, and enables to rebuild the Hilbert-Pólya operator, thus proving Riemann hypothesis.

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Competing Interests

Author has declared that no competing interests exist.

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