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Research Article

Approximate Analytical Solution to Nonlinear Delay Differential Equations by Using Sumudu Iterative Method

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In this study, an efficient analytical method called the Sumudu Iterative Method (SIM) is introduced to obtain the solutions for the nonlinear delay differential equation (NDDE). This technique is a mixture of the Sumudu transform method and the new iterative method. The Sumudu transform method is used in this approach to solve the equation's linear portion, and the new iterative method's successive iterative producers are used to solve the equation's nonlinear portion. Some basic properties and theorems which help us to solve the governing problem using the suggested approach are revised. The benefit of this approach is that it solves the equations directly and reliably, without the prerequisite for perturbations or linearization or extensive computer labor. Five sample instances from the DDEs are given to confirm the method's reliability and effectiveness, and the outcomes are compared with the exact solution with the assistance of tables and graphs after taking the sum of the first eight iterations of the approximate solution. Furthermore, the findings indicate that the recommended strategy is encouraging for solving other types of nonlinear delay differential equations.

1. Introduction

The more general kind of differential equations (DEs) is called functional differential equations (FDEs), as well as the delay differential equation (DDE) is the simplest and may be the most natural class of FDEs [1]. DDEs constitute a large and significant class of dynamical systems. Time delays are natural components of the dynamic processes of biology, ecology, physiology, economics, epidemiology, and mechanics, and so a genuine model of these processes must comprise time delays. DDEs are a form of DEs in which the derivative of the unknown function at a specific period is provided in terms of values of the function at past periods. Announcing delays in models enriches the vitality of these models and allows a perfect interpretation of actual occurrences [2–4]. DDEs arise frequently in various physical occurrences. To be precise, they are vital once ordinary differential equation- (ODE-) based models are ineffective. In disparate ODEs, where preliminary conditions are stated at the initial point, DDEs need the history of the system over the delayed intervals which are then provided as preliminary conditions. DDEs are infinite-dimensional and challenging to appraise since delay terms exist in the model [5]. Hence, solving DDEs is an important area of investigation for different scientists.

Recently, many researchers established and investigated various analytical and numerical methods to obtain approximate/exact solutions for nonlinear DEs as well as DDEs [6–8]. The variational iteration method (VIM) was employed by the authors of the publication [9–12] to discover a rough solution to nonlinear DDEs. To solve models of delayed vector-borne illnesses and delayed protein degradation, [13] incorporated the differential transform technique, Tarig transformation method, and Padè approximation in 2022. The Padè approximation is applied to this method to broaden the approximation solutions' convergence domain. Kumar and Methi [14] obtained the numerical result of some NDDEs with the support of Banach contraction method (BCM). To demonstrate that the suggested technique is fit for solving NDDEs, they also offered the numerical results, convergence theorem, and error analysis for various DDEs. In the publication [15], the authors debate an inverse problem for the NDDE that consists of approximating a solution by defining the beginning moment and delay parameter based on the experimental data. The inverse issue is seen as a nonlinear optimal control problem using the stated approach, for which the requisite criteria of optimality are established and shown. A method based on a better parallel evolutionary algorithm is used to solve the derived optimum control issue. In his publication [16], Familua proposes a straightforward method for solving specific second-order DDEs numerically without reducing them to systems of lower orders. Power series and an exponential function were combined to create the approaches utilizing a collocation approach. The first two grid points interpolate the approximation basis functions, and both grid and off-grid points collocate them. Additionally, the author combined the generated schemes and their derivatives to create block techniques, which enable the simultaneous direct solution of second-order DDEs without the laborious requirement of building separate predictors. To analyze the semianalytical solution of the DDEs, the writers of the effort [17] construct a novel technique called the Mohand homotopy perturbation transform method (MHPTM). This approach combines the Mohand transform with the homotopy perturbation method (HPTM). A novel class of neutral FDDEs using the generalized ψ-Caputo derivative on a partially ordered Banach space is investigated by the authors of the study [18]. The Dhage approach and the Banach contraction standard are used to demonstrate the presence and uniqueness of the solutions to the specified boundary value issue. The authors of the paper [19] considered a class of singularly perturbed advanced DDEs of convection-diffusion type. They use finite and hybrid difference schemes to solve the problem on piecewise Shishkin mesh. Readers can consult references [5, 20–24] for more work on DDEs that are linked to their interests.

Several intellectuals have created a variety of analytical integral transform techniques for precise and approximated answers during the past few years, including the Laplace transform method [25, 26], Shehu transform method [27, 28], Aboodh transform method [22], Sumudu transform method [29], Elzaki transform method [30], and Mohand transform method [17]. The Sumudu transform method is an integral like the Laplace transform method, introduced in the early 1990s by Watugala [31] to solve DEs and control engineering problems. This method converts a linear DE to an algebraic equation that can be resolved by consuming formal algebraic methods. The initial DE can then be resolved by employing the inverse Sumudu transformations. When matched to other methods, the Sumudu transform technique has the advantage of providing an accurate result quickly and without making any restricting assumptions about the result (see [29]). Unfortunately, this transform fails to crack nonlinear partial DEs, unlike the other integral transforms [32]. In these circumstances, the Sumudu transform method is frequently recycled with other well-used techniques such as the Adomian decomposition method (ADM), HPM, VIM, and the new iterative method [29, 33]. Alternatively, we may frequently employ integral transform techniques such as Laplace, Elzaki, and Shehu in place of the Sumudu transform method. Likewise, techniques like HPM, VIM, and ADM may be employed in place of the new iterative procedure. The HPM, also known as the He-Laplace technique, was developed by Mishra and Tripathi [34] and combines the Laplace transform with He's polynomials to solve DDEs, which have a variety of applications in physics, digital image processing, signal processing, and applied disciplines. With the usage of the Laplace transform homotopy perturbation approach (He-Laplace method), the writers of the study [35] were able to acquire the approximate analytical solutions of four distinct kinds of conformable partial DEs. We encourage readers to look at the descriptions of a few integral transforms in Table 1 [36].

Table 1: Some known integral transforms [17, 22, 36].

Integral transform	Expression
Elzaki transform	$E[f(t)] = u\int_0^\infty e^{-(t/u)} f(t)dt$
Natural transform	$N[f(t)] = \int_0^\infty e^{-(t/u)} f(st)dt$
Shehu transform	$S[f(t)] = \int_0^\infty e^{-(st/u)} f(t)dt$
Sawi transform	$S[f(t)] = 1/u^2\int_0^\infty e^{-(t/u)} f(t)dt$
Pourreza transform	$T[f(t)] = u\int_0^\infty e^{-u^2 t} f(t)dt$
Ara transform	$T[f(t)] = u\int_0^\infty e^{t^{n-1}} e^{-ut} f(t)dt$
Laplace transform	$L[f(t)] = \int_0^\infty e^{-ut} f(t)dt$
Sadik transform	$S[f(t)] = 1/u^{\beta}\int_0^\infty e^{-u^{\beta} t} f(t)dt$
Mohand transform	$M[f(t)] = u^2\int_0^\infty e^{-ut} f(t)dt$
Mahgoub transform	$M[f(t)] = u\int_0^\infty e^{-ut} f(t)dt$
Aboodh transform	$A[f(t)] = 1/u\int_0^\infty e^{-ut} f(t)dt$

The major goal of this study is to suggest the Sumudu Iterative Method (SIM), a trustworthy analytical technique for obtaining precise answers to NDDEs. This strategy combines the Sumudu transform method with an iterative method, two potent approaches. An iterative method (IM) has been presented by Daftardar-Gejji and Jafari to solve linear and nonlinear functional equations [37–40]. The IM has been effectively applied in many kinds of investigation to solve some linear and nonlinear PDEs and ODEs, NDDEs, higher-order integro-DEs, two-dimensional nonlinear Sine Gordon equation (NLSGE), and Korteweg-de Vries equations [25, 41–43]. The author of the study [29] successfully integrated the double Sumudu transform with the iterative approach to produce an approximate analytical answer to the one-dimensional coupled NLSGE, which cannot be solved by utilizing the double Sumudu transform alone. In this approach, the linear part of the problem was handled utilizing the double Sumudu transform method, and the nonlinear part through an additional iterative approach. According to Deresse [26], recent research has been done on the exact analytical solutions to the $(2+1)$-dimensional nonlinear conformable fractional telegraph equation utilizing the conformable fractional triple Laplace transform approach in blend with the innovative iterative method. In the specified technique, the new iterative method's consecutive iteration process is used to eliminate the equation's nonlinear noise components, and one iteration yields the exact answer. While the conformable fractional triple Laplace transform method is reused to resolve the linear constituent of the issue. To solve fractional-order Cauchy reaction-diffusion equations (CRDEs) in a vision of conformable derivative (CD), Rezapour et al. engaged the innovative iterative technique in aggregation with the Shehu transform scheme in their greatest latest publication [44].

The primary driving force behind the current study endeavor is the fact that the amalgamation of the Sumudu transform method and the new iterative SIM has not yet been investigated to solve NDDEs in the literature. The linear DDE was the subject of other research projects as well. This study work primarily focuses on NDDEs as a result, which is our second reason for introducing the novel approach SIM. The outcomes of the samples exhibit the accuracy and potency of this method, which does not require the imposition of any extra constraints to arrive at an analytical answer to the anticipated difficulties. It is a skilled scheme for reducing the number of calculations while keeping the answer more accurate and efficient.

In this work, we consider the following DDE in the form [22]

$$\frac{d^n y(t)}{dt^n} + p(y(t)) + N(y(t-\tau)) = f(t), \quad n = 1, 2, 3, \cdots, \tag{1}$$

subject to

$$y^{(k)}(0) = {y_0}^k, \tag{2}$$

where $d^n y/dt^n$ is the derivative of y of order n, P is the linear bounded operator, N is a nonlinear bounded operator, $f(t)$ is a given continuous function, and $y = y(t)$.

The remaining parts of this paper are structured as follows: Section 2 presents the basic explanations and properties of the Sumudu transform method. In Section 3, we clarify the methodology of the new iterative method. Section 4 demonstrates how SIM is pragmatic to the proposed problem DDE. Section 5 is dedicated to illustrating the SIM to five problems. In Section 6, we talk over the numerical results and illustrate the accuracy and efficiency of the SIM. Lastly, concluding remarks are outlined in Section 7.

2. Sumudu Transform Method

The definitions, characteristics, and theorems of the Sumudu transform scheme that we employed in this work are offered in this section (refer to [29, 45–52]).

Definition 1. For any real numbers $t > 0$, the function $G(u)$ is defined as the Sumudu transform of a function $f(t)$ which is given by

$$S\{f(t)\} = G(u) = \frac{1}{s}\int_0^\infty e^{-(t/u)} f(t)dt = \lim_{\tau \longrightarrow \infty}\int_0^\tau e^{-(t/u)} f(t)dt. \tag{3}$$

Definition 2. The inverse Sumudu transform of a function $G(u)$ is denoted by the symbol $S^{-1}[G(u)] = f(t)$ and is defined by the Bromwich contour integral

$$S^{-1}[G(u)] = f(t) = \lim_{\beta \longrightarrow \infty} \frac{1}{2\pi i}\int_{\alpha - i\beta}^{\alpha + i\beta} e^{(t/u)} G(u)du. \tag{4}$$

2.1. Convergence Theorem of Sumudu Transform

Lemma 3. *Let $f(t)$ be a continuous function. If the integral $1/u\int_0^\infty e^{(-t/u)} f(t)dt$ converges at $u = u_0$, then the integral converges for $u < u_0$.*

Theorem 4. *Let $n \geq 1$ and let $G_n(u)$ be the Sumudu transforms of the function $f^{(n)}(t)$, then*

$$\begin{cases} G_1(u) = S\left[f'(t)\right] = \dfrac{G(u) - f(0)}{u} = \dfrac{G(u)}{u} - \dfrac{f(0)}{u}, \\ G_2(u) = S[f(t)] = \dfrac{G(u) - f(0)}{u^2} - \dfrac{f'(0)}{u} = \dfrac{G(u)}{u^2} - \dfrac{f(0)}{u^2} - \dfrac{f'(0)}{u}, \\ . \\ . \\ . \\ G_n(u) = S\left[f^{(n)}(t)\right] = \dfrac{G(u)}{u^n} - \dfrac{f(0)}{u^n} - -\dfrac{f'(0)}{u^{n-1}} - \cdots - \dfrac{f^{(n-1)}(0)}{u} = \dfrac{G(u)}{u^n} - \displaystyle\sum_{k=0}^{n-1} \dfrac{f^k(0)}{u^{n-k}}. \end{cases} \tag{5}$$

2.2. Existence and Uniqueness of the Sumudu Transform Method

Theorem 5 (existence). *If f is an exponential order, then its Sumudu transform $S[f(t)\,;u] = G(u)$ is given by*

$$G(u) = \frac{1}{u}\int_{-\infty}^\infty e^{(-t/u)} f(t)dt, \tag{6}$$

where $1/u = (1/\eta) + (i/\tau)$.

The defining integral for G exists at points $1/u = (1/\eta) + (i/\tau)$ in right half plane $1/\eta > 1/k$ and $1/\zeta > 1/L$.

Theorem 6 (uniqueness). *Let $f(t)$ and $g(t)$ be continuous functions defined for $t \geq 0$ which have Sumudu transforms, $F(u)$ and $G(u)$, respectively. If $F(u) = G(u)$ almost everywhere, then $f(t) = g(t)$, where u is a complex number.*

See [45, 46] for the proof.

2.3. Some Properties of Sumudu Transform

Property 7. Over the set of function A: $A=\{(f(t)|\exists M, \tau_1, \tau_2>0, |f(t)|<Me^{|t|\tau_j}, \text{if } t\in(-1)^j\times[0,\infty))\}$, the Sumudu transform is defined by

$$G(u)=S(f(t))=\int_0^\infty f(uat)e^{-t}dt, \quad u\in(-\tau_1,\tau_2). \tag{7}$$

Property 8 (linearity property). Let $f(t)$ and $g(t)$ be any two functions whose Sumudu transforms exist. Then, for arbitrary constant a and b, we have

$$S\{af(t)+bg(t)\}=aS\{f(t)\}+bS\{g(t)\}. \tag{8}$$

Proof. Using the definition of Sumudu transform and property of integral, we have

$$\begin{aligned} S[af(t)+bg(t)] &= \int_0^\infty [af(ut)+bg(ut)]e^{-t}dt \\ &= a\int_0^\infty f(ut)e^{-t}dt + bg(ut)e^{-t}dt \\ &= aS[f(t)]+bS[g(t)]. \end{aligned} \tag{9}$$

Hence, the proof. □

Property 9 (first scale preserving theorem). If $S[f(t)]=G(u)$, then $S[f(at)]=G(at)$, where a is nonzero constant.

Proof. Let $S[f(t)]=G(u)=\int_0^\infty f(ut)e^{-t}dt$, then

$$S[f(at)]=\int_0^\infty f(uat)e^{-t}dt=G(au). \tag{10}$$

□

Property 10 (first shifting theorem). If $S[f(t)]=G(u)$, then

$$S\left[f(t)e^{at}\right]=\frac{1}{1-au}G\left(\frac{u}{1-au}\right). \tag{11}$$

Property 11 (second shifting theorem). If $S[f'(t)]=G'(u)$, then $S[tf'(t)]=uG'(u)$.

3. The New Iterative Method (NIM)

Daftardar-Gejji and Jafari [38] have suggested a new iterative method (NIM) for resolving nonlinear functional equations of the form

$$y=f+N(y), \tag{12}$$

where f is a known function and N a nonlinear operator. In this scheme, we assume that equation (12) has a series answer of the form

$$y=\sum_{n=0}^{\infty} y_n(t). \tag{13}$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{n=0}^{\infty} y_n\right)=N(y_0)+\sum_{n=1}^{\infty}\left\{N\left(\sum_{n=0}^{k} y_n\right)-N\left(\sum_{n=0}^{k-1} y_n\right)\right\}, k=1,2,3,\cdots \tag{14}$$

This implies that

$$N(y)=N(y_0)+[N(y_0+y_1)-N(y_0)]+[N(y_0+y_1+y_2)-N(y_0+y_1)]+.\cdots \tag{15}$$

Using equations (14) and (13) in equation (12), we obtain

$$N\left(\sum_{n=0}^{\infty} y_n\right)=f+N(y_0)+\sum_{n=1}^{\infty}\left\{N\left(\sum_{n=0}^{k} y_n\right)-N\left(\sum_{n=0}^{k-1} y_n\right)\right\}. \tag{16}$$

From equation (16), we define the subsequent recurrence relation:

$$\begin{cases} y_0=f, \\ y_1=N(y_0), \\ y_2=N(y_0+y_1)-N(y_0), \\ y_3=N(y_0+y_1+y_2)-N(y_0+y_1), \\ . \\ . \\ . \\ y_n=N(y_0+y_1+y_2+.\cdots+y_n)-N(y_0+y_1+y_2+\cdots+y_{n-1}). \end{cases} \tag{17}$$

As a result, equation (12) is equivalent to

$$\sum_{n=0}^{\infty} y_n(t)=f+\sum_{n=1}^{\infty} y_n(t). \tag{18}$$

Therefore, the k^{th} term approximate answer of equation (1) is given by $y(t)=\sum_{n=0}^{k-1}y_n(t)=y_0+y_1+y_2+y_3+\cdots+y_{k-1}=f+N(y_0)+[N(y_0+y_1)-N(y_0)]+[N(y_0+y_1+y_2)-N(y_0+y_1)]+.\cdots$ (20)

3.1. Convergence of the New Iterative Method. This subsection offers the conditions for the convergence of the series (20). For more details, attracted readers may refer to [53].

Theorem 12. *If N is a continuously differentiable functional in a neighborhood of y_0 and $\|N^{(n)}(y_0)\|=Sup\{N^{(n)}(y_0)(h_1, h_2,\cdots,h_n): \|h_i\|\le 1, 1\le i\le n\}\le L$, for each "n" and for some real $L>0$ and $\|y_i\|\le M<1/e, i=1,2,3,\cdots$, then the series $\sum_{i=0}^{\infty}y_{i+1}$ is absolutely convergent and moreover,*

$$\|y_{i+1}\| \le LM^n e^{n-1}(e-1), \quad n = 1, 2, \cdots \tag{19}$$

Theorem 13. *If N is a continuously differentiable functional in a neighborhood of y_0 and $\|N^{(n)}(y_0)\| \le M \le 1/e$ for all n, then the series $\sum_{i=0}^{\infty} y_{i+1}$ is absolutely convergent.*

4. Sumudu Iterative Method (SIM)

In this section, the general form of the n^{th}-order DDE (1) with the initial value (2) is treated using the suggested method SIM.

Step 1. Applying the Sumudu transform on both sides of equation (1), we get

$$S\left[\frac{d^n y}{dt^n}\right] + S[P(y)] + S[N(y(t-\tau))] = S[f(t)]. \tag{20}$$

Step 2. Theorem4is used to obtain

$$\frac{S(y(t))}{u^n} - \frac{C}{u^{n-k}} + S[P(y)] + S[N(y(t-\tau))] = S[f(t)], \tag{21}$$

where $C = \sum_{k=0}^{n-1} f^k(0)$.(24)

Replacing (24) into (21) and shortening, we obtain

$$S[y(t)] = u^k C - u^n S[P(y)] + u^n S[f(t)] - u^n S[N(y_n(t-\tau))]. \tag{22}$$

Step 3. Applying the inverse Sumudu transform on both sides of equation (22), we get

$$y(t) = S^{-1}\left[u^k C + u^n S(f(t)) - u^n S(p(y))\right] - S^{-1}[u^n S(N(y_n(t-\tau)))]. \tag{23}$$

Step 4. Now, we surprise the iterative method. We assume that

$y(t) = \sum_{n=0}^{\infty} y_n(t)$ is the solution of equation (1).

Step 5. Substituting equation (27) into equation (23), we acquire

$$\sum_{n=0}^{\infty} y_n(t) = S^{-1}\left[u^k C + u^n S(f(t)) - u^n S(p(y))\right] - S^{-1}\left[u^n S\left(N\left(\sum_{n=0}^{\infty} y_n(t-\tau)\right)\right)\right]. \tag{24}$$

Step 6. Using (16), decompose the nonlinear term$N\left(\sum_{n=0}^{\infty} y_n(t-\tau)\right)$in equation (24) as follows:

$$\sum_{n=0}^{\infty} N(y(t-\tau)) = N(y_0) + \sum_{n=1}^{\infty}\left\{N\left(\sum_{n=0}^{k} y_n\right) - N\left(\sum_{n=0}^{k-1} y_n\right)\right\}. \tag{25}$$

Step 7. Replacing equation (25) in equation (24), we get

$$\sum_{n=0}^{\infty} y_n(t) = S^{-1}\left[u^k C + u^n S(f(t)) - u^n S(p(y))\right] - S^{-1}\left[u^n S\left(N(y_0) + \sum_{n=1}^{\infty}\left\{N\left(\sum_{n=0}^{k} y_n\right) - N\left(\sum_{n=0}^{k-1} y_n\right)\right\}\right)\right]. \tag{26}$$

Step 8. Define the recurrence relation from the systems of equation (26) as follows:

$$\begin{cases} y_0(t) = S^{-1}\left[u^k C + u^n S(f(t)) - u^n S(p(y))\right], \\ y_1(t) = -S^{-1}[u^n S(N(y_0))], \\ \cdot \\ \cdot \\ \cdot \\ y_{n+1}(t) = -S^{-1}\left[u^n S\left(\sum_{n=1}^{\infty}\left\{N\left(\sum_{n=0}^{k} y_n\right) - N\left(\sum_{n=0}^{k-1} y_n\right)\right\}\right)\right], \quad n \ge 1. \end{cases} \tag{27}$$

Therefore, in truncated series form, the approximate analytical solution of the DDE (1) is given by

$$y(t) = \lim_{N \to \infty} \sum_{m=0}^{N} y_m(t) = y_0 + y_1 + y_2 + y_3 + \cdots \tag{28}$$

5. Illustrative Examples

In this part, four nonlinear DDEs are used to demonstrate the effectiveness and validity of the SIM.

Example 1 (see [44]). Consider a first-order nonlinear DDE:

$$\frac{d}{dt} y(t) = 1 - 2y^2\left(\frac{t}{2}\right), \tag{29}$$

subject to the initial condition

$$y(0) = 0. \tag{30}$$

The analytical solution is given by $y(t) = \sin t$..

Solution: taking the Sumudu transform on both sides of equation (29), we get

$$\frac{G(u)}{u} - \frac{y(0)}{u} = S[1] - S\left[2y^2\left(\frac{t}{2}\right)\right]. \tag{31}$$

Taking the place of equation (30) into equation (29), we achieve

$$G(u) = u - uS\left[2y^2\left(\frac{t}{2}\right)\right]. \tag{32}$$

Subsequently, taking the inverse Sumudu transform of equation (32) implies that

$$S^{-1}[G(u)] = S^{-1}[u] - S^{-1}\left[uS\left[2y^2\left(\frac{t}{2}\right)\right]\right]. \tag{33}$$

By using the properties of Sumudu transforms listed in Table 2, equation (33) becomes

$$y(t) = t - S^{-1}\left[uS\left[2y^2\left(\frac{t}{2}\right)\right]\right]. \tag{34}$$

Now, applying the new iterative method to equation (34) in a vision of equations (27)–(31), we attain the components of the solution as follows:

$$\begin{aligned} y_0(t) &= t, \\ y_0\left(\frac{t}{2}\right) &= \frac{t}{2}, \\ y_1(t) &= -S^{-1}\left[uS\left[2{y_0}^2\left(\frac{t}{2}\right)\right]\right] = -2S^{-1}\left[uS\left(\frac{t^2}{4}\right)\right] \\ &= -S^{-1}\left[u\left(u^2\right)\right] = -S^{-1}\left(u^3\right) = -\frac{t^3}{6} = -\frac{t^3}{3}!. \end{aligned} \tag{35}$$

Hence, $y_1(t) = -(t^3/3!)$ and $y_1(t/2) = -(t^3/48)$.

$$\begin{aligned} y_2(t) &= N\left[y_0\left(\frac{t}{2}\right) + y_1\left(\frac{t}{2}\right)\right] - N\left[y_0\left(\frac{t}{2}\right)\right] \\ &= -2S^{-1}\left[uS\left[\left(\frac{t}{2} - \frac{t^3}{48}\right)^2 - \left(\frac{t}{2}\right)^2\right]\right] \\ &= -2S^{-1}\left[uS\left[\frac{t^2}{4} - \frac{t^4}{48} + \frac{t^6}{2{,}304} - \frac{t^2}{4}\right]\right] \\ &= -2S^{-1}\left[uS\left[-\frac{t^4}{48} + \frac{t^6}{2{,}304}\right]\right] = -S^{-1}\left[u\left(-u^4 + \frac{6!u^6}{1152}\right)\right]. \end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned} y_2(t) &= -S^{-1}\left[-u^5 + \frac{6!u^7}{1152}\right] = \frac{t^5}{5!} - \frac{t^7}{8064} \& y_2\left(\frac{t}{2}\right) = \frac{t^5}{3840} - \frac{t^7}{1032192}, \\ y_3(t) &= N\left[y_0\left(\frac{t}{2}\right) + y_1\left(\frac{t}{2}\right) + y_2\left(\frac{t}{2}\right)\right] \\ &\quad - N\left[y_0\left(\frac{t}{2}\right) + y_1\left(\frac{t}{2}\right)\right] \\ &= -2S^{-1}\left[uS\left[\left(\frac{t}{2} - \frac{t^3}{48} + \frac{t^5}{3840} - \frac{t^7}{1032192}\right)^2 - \left(\frac{t}{2} - \frac{t^3}{48}\right)^2\right]\right] \\ &= -2S^{-1}\left[uS\left[\frac{t^6}{3840} - \frac{12.2t^8}{1032192} + \frac{2.68t^{10}}{24772608} - \frac{t^{12}}{1981808640} + \frac{t^{14}}{1065420324864}\right]\right] \\ &= -\frac{t^7}{1920} + \frac{6.1t^9}{258048} - \frac{0.64t^{11}}{3096576} + \frac{t^{13}}{990904320} - \frac{t^{15}}{532710162432}. \end{aligned} \tag{37}$$

Table 2: Standard Sumudu transform for some special functions [52].

$f(t)$	$G(u) = S(f(t))$
1	1
t^n	$n!u^n$
$t^n e^{at}$	$n!u^n/(1-au)^{n+1}$
$\sin at$	$au/1 + a^2u^2$
$\cos at$	$1/1 + a^2u^2$
$\sinh at$	$au/1 - a^2u^2$
$\cosh at$	$1/1 - a^2u^2$
$e^{ct}f(at)$	$(1/1-cu)G(au/1-cu)$
$H(t-a)$	$e^{-a/u}$
$H(t-a)f(t-a)$	$e^{-a/u}G(u)$

Likewise, we can obtain the leftover values by means of the recurrence relation $y_{n+1}(t) = -2S^{-1}[uS(\sum_{n=1}^{\infty}\{N(\sum_{n=0}^{k}{y_n}^2(t/2)) - N(\sum_{n=0}^{k-1}{y_n}^2(t/2))\})]$, $n \geq 1$..

Now, in vision of (28), the solution of Example 1 is

$$\begin{aligned} y(t) &= y_0(t) + y_1(t) + y_2(t) + \cdots = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{1.6t^7}{4032} + \frac{6.1t^9}{258048} \\ &\quad - \frac{0.64t^{11}}{3096576} + \frac{t^{13}}{990904320} - \frac{t^{15}}{532710162432} +. \cdots \end{aligned} \tag{38}$$

Example 2. Consider the following nonlinear second-order DDE [23]:

$$\frac{d^2y(t)}{dt^2} = -y(t) + 5y^2\left(\frac{t}{2}\right), \quad t \geq 0, \tag{39}$$

with initial condition

$$\begin{aligned} y(0) &= 1, \\ \frac{dy(0)}{dt} &= -2. \end{aligned} \tag{40}$$

The analytical solution is given by $y(t) = e^{-2t}$..

Solution: taking the Sumudu transform on both sides of equation (39), we get

$$\frac{G(u) - y(0)}{u^2} - \frac{y'(0)}{u} = S\left[-y(t) + 5y^2\left(\frac{t}{2}\right)\right]. \tag{41}$$

Replacing equation (40) into equation (39), we get

$$G(u) = 1 - 2u + u^2 S\left[-y(t) + 5y^2\left(\frac{t}{2}\right)\right]. \tag{42}$$

Subsequently, taking the inverse Sumudu transform of equation (42) suggests that

$$S^{-1}[G(u)] = S^{-1}\left[1 - 2u + u^2 S\left[-y(t) + 5y^2\left(\frac{t}{2}\right)\right]\right]. \quad (43)$$

By using the properties of Sumudu transforms listed in Table 2, equation (43) becomes

$$y(t) = 1 - 2t + S^{-1}\left[u^2 S\left[-y(t) + 5y^2\left(\frac{t}{2}\right)\right]\right]. \quad (44)$$

Now, applying the new iterative method to equation (44) in the opinion of equations (27)–(31), we achieve the components of the answer as follows:

$$\begin{aligned} y_0(t) &= 1 - 2t, \\ y_0\left(\frac{t}{2}\right) &= 1 - 2\left(\frac{t}{2}\right) = 1 - t, \\ y_1(t) &= S^{-1}\left[u^2 S\left[-y_0(t) + 5{y_0}^2\left(\frac{t}{2}\right)\right]\right] \\ &= S^{-1}\left[u^2 S\left[-(1-2t) + 5\left[(1-t)^2\right]\right]\right] \\ &= S^{-1}\left[u^2\left[-1 + 2u + 5\left[1 - 2u + 2u^2\right]\right]\right] \\ &= S^{-1}\left[-u^2 + 2u^3\right] + 5S^{-1}\left[u^2 - 2u^3 + 2u^4\right]. \end{aligned} \quad (45)$$

As a result,

$$\begin{aligned} y_1(t) &= \frac{-t^2}{2} + \frac{t^3}{3} + 5\left[\frac{t^2}{2} - \frac{t^3}{3} + \frac{t^4}{12}\right], \\ y_1\left(\frac{t}{2}\right) &= \frac{t^2}{2} - \frac{t^3}{3} + \frac{5t^4}{192} . y_2(t) = -y_1(t) + N\left[y_0\left(\frac{t}{2}\right) + y_1\left(\frac{t}{2}\right)\right] - N\left[y_0\left(\frac{t}{2}\right)\right], \\ y_2(t) &= S^{-1}\left[u^2 S[-y_1(t)]\right] + 5S^{-1}\left[u^2 S\left[\left(1 - t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{5t^4}{192}\right)^2 - (1-t)^2\right]\right] = S^{-1}\left[u^2 S\left[-\left(2t^2 - \frac{4t^3}{3} + \frac{5t^4}{12}\right)\right]\right] \\ &\quad + 5S^{-1}\left[u^2 S\left(t^2 - \frac{2t^3}{3} + \frac{29t^4}{96} - \frac{t^5}{3} + \frac{79t^6}{576} - \frac{10t^7}{576} + \frac{25t^8}{36864}\right)\right] \Rightarrow y_2(t) = \frac{-t^4}{6} + \frac{t^5}{15} - \frac{t^6}{72} \\ &\quad + 5\left(\frac{t^4}{12} - \frac{t^5}{12} + \frac{29t^6}{2880} - \frac{t^7}{126} + \frac{79t^8}{32256} - \frac{25t^9}{10368} + \frac{10t^{10}}{1327104}\right), \\ y_2\left(\frac{t}{2}\right) &= \frac{-t^4}{96} + \frac{t^5}{480} - \frac{t^6}{4608} + 5\left(\frac{t^4}{192} - \frac{t^5}{384} + \frac{29t^6}{184320} - \frac{t^7}{16128} + \frac{79t^8}{8257536} - \frac{25t^9}{5308416} + \frac{10t^{10}}{1358954496}\right). \\ y_3(t) &= -y_2(t) + N\left[y_0\left(\frac{t}{2}\right) + y_1\left(\frac{t}{2}\right) + y_2\left(\frac{t}{2}\right)\right] - N\left[y_0\left(\frac{t}{2}\right) + y_1\left(\frac{t}{2}\right)\right] \\ &= S^{-1}\left[u^2 S\left[-\left(\frac{-t^4}{6} + \frac{t^5}{15} - \frac{t^6}{72} + 5\left(\frac{t^4}{12} - \frac{t^5}{12} + \frac{29t^6}{2880} - \frac{t^7}{126} + \frac{79t^8}{32256} - \frac{25t^9}{10368} + \frac{10t^{10}}{1327104}\right)\right)\right]\right] \\ &\quad + 5S^{-1}\left[u^2 S\left[\left(1 - t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{5t^4}{192} + \frac{-t^4}{96} + \frac{t^5}{480} - \frac{t^6}{4608} + 5\left(\frac{t^4}{192} - \frac{t^5}{384} + \frac{29t^6}{184320} - \frac{t^7}{16128} + \frac{79t^8}{8257536} - \frac{25t^9}{5308416} + \frac{10t^{10}}{1358954496}\right)\right)^2 - \left(1 - t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{5t^4}{192}\right)^2\right]\right]. \end{aligned} \quad (46)$$

Similarly, we can obtain the continuing values by using the recurrence relation.

$$y_{n+1}(t) = S^{-1}\left[u^2 S(-y_n(t))\right] + 5S^{-1}\left[u^2 S\left(\sum_{n=1}^{\infty}\left\{N\left(\sum_{n=0}^{k} {y_n}^2\left(\frac{t}{2}\right)\right) - N\left(\sum_{n=0}^{k-1} {y_n}^2\left(\frac{t}{2}\right)\right)\right\}\right)\right], \quad n \geq 1. \tag{47}$$

Now, in sight of (28), the solution of Example 2 is

$$\begin{aligned} y(t) &= y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots = 1 - 2t + 2t^2 - \frac{4t^3}{3} \\ &+ \frac{2t^4}{3} - \frac{7t^5}{20} + \frac{21t^6}{576} - \frac{5t^7}{126} + \frac{395t^8}{32256} - \frac{125t^9}{10368} \\ &+ \frac{50t^{10}}{1327104} + . \cdots \end{aligned} \tag{48}$$

Example 3. Consider the following nonlinear second-order DDE [24]:

$$\frac{d^2 y(t)}{dt^2} = y^2\left(\frac{t}{2}\right), \quad t \geq 0, \tag{49}$$

with initial condition

$$\begin{aligned} y(0) &= 1, \\ \frac{dy(0)}{dt} &= 1. \end{aligned} \tag{50}$$

The analytical solution is given by $y(t) = e^t$.

Solution: taking the Sumudu transform on both sides of equation (49), we acquire

$$\frac{G(u) - y(0)}{u^2} - \frac{y'(0)}{u} = S\left[y^2\left(\frac{t}{2}\right)\right]. \tag{51}$$

Replacing equation (50) into equation (49), we get

$$G(u) = 1 + u + u^2 S\left[y^2\left(\frac{t}{2}\right)\right]. \tag{52}$$

Consequently, taking the inverse Sumudu transform of equation (52) implies that

$$S^{-1}[G(u)] = S^{-1}\left[1 + u + u^2 S\left[y^2\left(\frac{t}{2}\right)\right]\right]. \tag{53}$$

By using the properties of Sumudu transforms listed in Table 2, equation (53) becomes

$$y(t) = 1 + t + S^{-1}\left[u^2 S\left[y^2\left(\frac{t}{2}\right)\right]\right]. \tag{54}$$

Now, applying the new iterative method to equation (54) in the understanding of equations (27)–(31), we attain the components of the answer as follows:

$$\begin{aligned} y_0(t) &= 1 + t, \\ y_0\left(\frac{t}{2}\right) &= 1 + \frac{t}{2}, \\ y_1(t) &= S^{-1}\left[u^2 S\left[{y_0}^2\left(\frac{t}{2}\right)\right]\right], \\ y_1(t) &= S^{-1}\left[u^2 S\left[\left(1 + \frac{t}{2}\right)^2\right]\right] = S^{-1}\left[u^2 S\left[1 + t + \frac{t^2}{4}\right]\right] \\ &= S^{-1}\left[u^2\left(1 + u + \frac{u^2}{2}\right)\right] = S^{-1}\left[u^2 + u^3 + \frac{u^4}{2}\right] \\ &= \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{48} \text{ and } y_1\left(\frac{t}{2}\right) = \frac{t^2}{8} + \frac{t^3}{48} + \frac{t^4}{768}, \end{aligned}$$

$$\begin{aligned} y_2(t) &= N\left[y_0\left(\frac{t}{2}\right) + y_1\left(\frac{t}{2}\right)\right] - N\left[y_0\left(\frac{t}{2}\right)\right] \\ &= S^{-1}\left[u^2 S\left[\left(1 + \frac{t}{2} + \frac{t^2}{8} + \frac{t^3}{48} + \frac{t^4}{768}\right)^2 - \left(1 + \frac{t}{2}\right)^2\right]\right] \\ &= S^{-1}\left[u^2 S\left[\frac{t^2}{4} + \frac{t^3}{6} + \frac{5t^4}{128} + \frac{5t^5}{768} + \frac{7t^6}{9216} + \frac{t^7}{18432} \right.\right. \\ &\left.\left. + \frac{t^8}{589824}\right]\right] = S^{-1}\left[u^2\left[\frac{u^2}{2} + u^3 + \frac{15u^4}{16} + \frac{25u^5}{32} \right.\right. \\ &\left.\left. + \frac{35u^6}{64} + \frac{35u^7}{128} + \frac{35u^8}{512}\right]\right] = S^{-1}\left[\frac{u^4}{2} + u^5 + \frac{15u^6}{16} \right. \\ &\left. + \frac{25u^7}{32} + \frac{35u^8}{64} + \frac{35u^9}{128} + \frac{35u^{10}}{512}\right] = \frac{t^4}{48} + \frac{t^5}{120} \\ &+ \frac{t^6}{768} + \frac{10t^7}{64512} + \frac{t^8}{73728} + \frac{t^9}{1327104} + \frac{t^{10}}{53084160}, \end{aligned}$$

$$\begin{aligned} y_2\left(\frac{t}{2}\right) &= \frac{t^4}{768} + \frac{t^5}{3840} + \frac{t^6}{49152} + \frac{10t^7}{8257536} + \frac{t^8}{18874368} \\ &+ \frac{t^9}{679477248} + \frac{t^{10}}{54358180000} \\ &= S^{-1}\left[u^2 S\left[\left(1 + \frac{t}{2} + \frac{t^2}{8} + \frac{t^3}{48} + \frac{t^4}{768} + \frac{t^4}{768} + \frac{t^5}{3840} + \frac{t^6}{49152} + \frac{10t^7}{8257536} + \frac{t^8}{18874368} + \frac{t^9}{679477248} + \frac{t^{10}}{54358180000}\right)^2 \right.\right. \\ &\left.\left. - \left(1 + \frac{t}{2} + \frac{t^2}{8} + \frac{t^3}{48} + \frac{t^4}{768}\right)^2\right]\right]. \end{aligned} \tag{55}$$

Similarly, we can get the continuing values by consuming the recurrence relation

$$y_{n+1}(t) = S^{-1}\left[u^2 S\left(\sum_{n=1}^{\infty}\left\{N\left(\sum_{n=0}^{k} {y_n}^2\left(\frac{t}{2}\right)\right) - N\left(\sum_{n=0}^{k-1} {y_n}^2\left(\frac{t}{2}\right)\right)\right\}\right)\right], \quad n \geq 1. \tag{56}$$

Now, in vision of (28), the solution of Example 3 is

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{768} + \frac{10t^7}{64512} + \frac{t^8}{73728} + \frac{t^9}{1327104} + \frac{t^{10}}{53084160} +. \cdots \quad (57)$$

Example 4. Consider the following nonlinear first-order DDE [24]:

$$\frac{d}{dt}y(t) - 2ty^4\left(\frac{t}{2}\right) = 0, \quad t \geq 0, \quad (58)$$

with initial condition

$$y(0) = 1. \quad (59)$$

The analytical solution is given by $y(t) = e^{t^2}$.

Solution: taking the Sumudu transform on both sides of equation (58), we get

$$\frac{G(u) - y(0)}{u} = S\left[2ty^4\left(\frac{t}{2}\right)\right]. \quad (60)$$

Substituting equation (59) into equation (58), we get

$$G(u) = 1 + uS\left[2ty^4\left(\frac{t}{2}\right)\right]. \quad (61)$$

Subsequently, taking the inverse Sumudu transform of equation (61) implies that

$$S^{-1}[G(u)] = S^{-1}\left[1 + uS\left[2ty^4\left(\frac{t}{2}\right)\right]\right]. \quad (62)$$

By using the properties of Sumudu transforms listed in Table 2, equation (62) becomes

$$y(t) = 1 + S^{-1}\left[uS\left[2ty^4\left(\frac{t}{2}\right)\right]\right]. \quad (63)$$

Now, applying the new iterative method to equation (63) in the opinion of equations (27)–(31), we acquire the components of the solution as follows:

$$y_0(t) = 1,$$
$$y_0\left(\frac{t}{2}\right) = 1,$$

$$y_1(t) = S^{-1}\left[uS\left[2t{y_0}^4\left(\frac{t}{2}\right)\right]\right] = S^{-1}\left[uS\left[2t(1)^4\right]\right] = S^{-1}[uS[2t]] = S^{-1}[u(2u)] = S^{-1}\left[2u^2\right] = t^2. \quad (64)$$

Hence, $y_1(t) = t^2$ and $y_1(t/2) = t^2/4$.

$$y_2(t) = N\left[y_0\left(\frac{t}{2}\right) + y_1\left(\frac{t}{2}\right)\right] - N\left[y_0\left(\frac{t}{2}\right)\right],$$

$$\begin{aligned} y_2(t) &= S^{-1}\left[uS\left[2t\left(\left(1 + \frac{t^2}{4}\right)^4 - (1)^4\right)\right]\right] \\ &= S^{-1}\left[uS\left[2t\left(1 + t^2 + \frac{3t^4}{8} + \frac{t^6}{16} + \frac{t^8}{256} - (1)\right)\right]\right] \\ &= S^{-1}\left[uS\left[2t\left(t^2 + \frac{3t^4}{8} + \frac{t^6}{16} + \frac{t^8}{256}\right)\right]\right] \\ &= S^{-1}\left[uS\left[2t^3 + \frac{3t^5}{4} + \frac{t^7}{8} + \frac{t^9}{128}\right]\right] \\ &= S^{-1}\left[u\left[12u^3 + 90u^5 + 630u^7 + 2835u^9\right]\right] \\ &= S^{-1}\left[12u^4 + 90u^6 + 630u^8 + 2835u^{10}\right] \\ &= \frac{t^4}{2} + \frac{t^6}{8} + \frac{t^8}{64} + \frac{t^{10}}{1280}. \end{aligned} \quad (65)$$

Therefore,

$$y_2\left(\frac{t}{2}\right) = \frac{t^4}{32} + \frac{t^6}{512} + \frac{t^8}{16384} + \frac{t^{10}}{1310720},$$

$$y_3(t) = N\left[y_0\left(\frac{t}{2}\right) + y_1\left(\frac{t}{2}\right) + y_2\left(\frac{t}{2}\right)\right] - N\left[y_0\left(\frac{t}{2}\right) + y_1\left(\frac{t}{2}\right)\right],$$

$$y_3(t) = S^{-1}\left[uS\left[2t\left(\left(1 + \frac{t^2}{4} + \frac{t^4}{32} + \frac{t^6}{512} + \frac{t^8}{16384} + \frac{t^{10}}{1310720}\right)^4 - \left(1 + \frac{t^2}{4}\right)^4\right)\right]\right]. \quad (66)$$

Similarly, we can gain the remaining values by using the recurrence relation

$$y_{n+1}(t) = S^{-1}\left[uS\left(\sum_{n=1}^{\infty}\left\{N\left(\sum_{n=0}^{k}{y_n}^2\left(\frac{t}{2}\right)\right) - N\left(\sum_{n=0}^{k-1}{y_n}^2\left(\frac{t}{2}\right)\right)\right\}\right)\right], \quad n \geq 2. \quad (67)$$

Now, in the understanding of (28), the solution of Example 4 is

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots = 1 + t^2 + \frac{t^4}{2} + \frac{t^6}{8} + \frac{t^8}{64} + \frac{t^{10}}{1280} +. \cdots \quad (68)$$

Example 5. Consider the nonlinear proportional delay differential equation [(39)]:

$$\frac{d^2y(t)}{dt^2} = 1 - 2y^2\left(\frac{t}{2}\right), \quad 0 < t < 1, \quad (69)$$

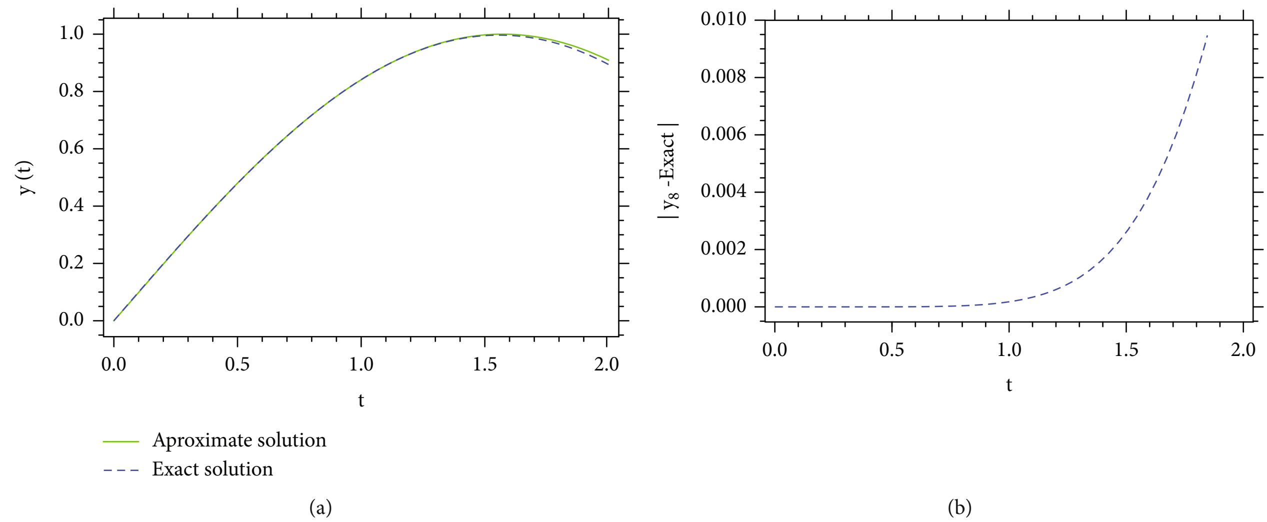


Figure 1: (a) 2D solution plots of Example 1 obtained by the present method in comparison with the exact solutions. (b) The absolute errors of Example 1.

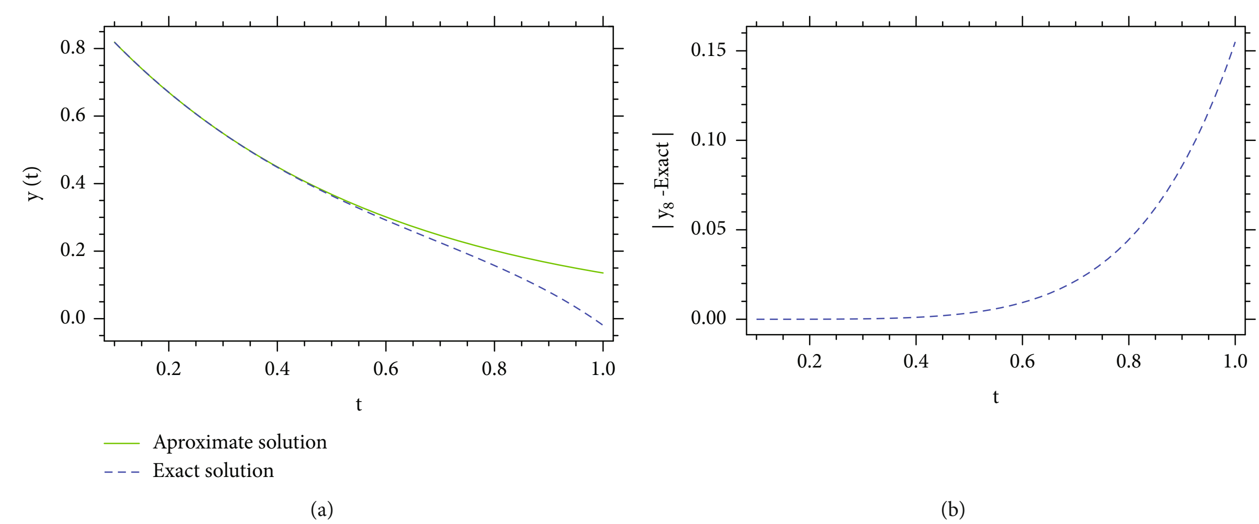


Figure 2: (a) 2D solution plots of Example 2 obtained by the present method in comparison with the exact solutions. (b) The absolute errors of Example 2.

with initial condition

$$y(0) = 1, \quad \frac{dy(0)}{dt} = 0. \tag{70}$$

The exact solution is given by

$$y(t) = \cos t. \tag{71}$$

Solution: taking the Sumudu transform on both sides of equation (69), we get

$$\frac{G(u) - y(0)}{u^2} - \frac{y'(0)}{u} = S[1] - S\left[2y^2\left(\frac{t}{2}\right)\right]. \tag{72}$$

Replacing equation (70) into equation (72), we get

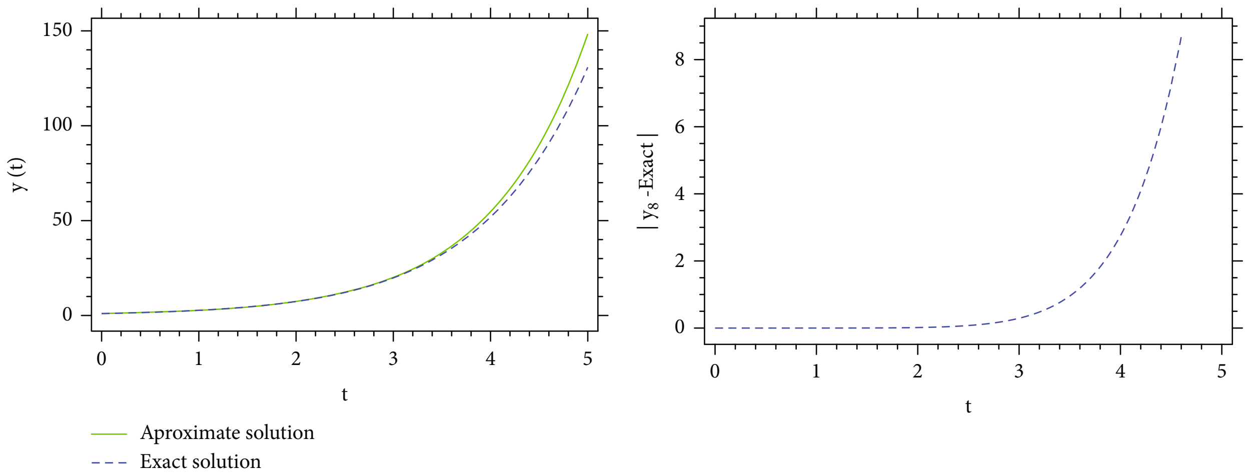
$$G(u) = 1 + u^2 - u^2 S\left[2y^2\left(\frac{t}{2}\right)\right]. \tag{73}$$

Consequently, taking the inverse Sumudu transform of equation (73) implies that

$$S^{-1}[G(u)] = S^{-1}\left[1 + u^2 - u^2 S\left[2y^2\left(\frac{t}{2}\right)\right]\right]. \tag{74}$$

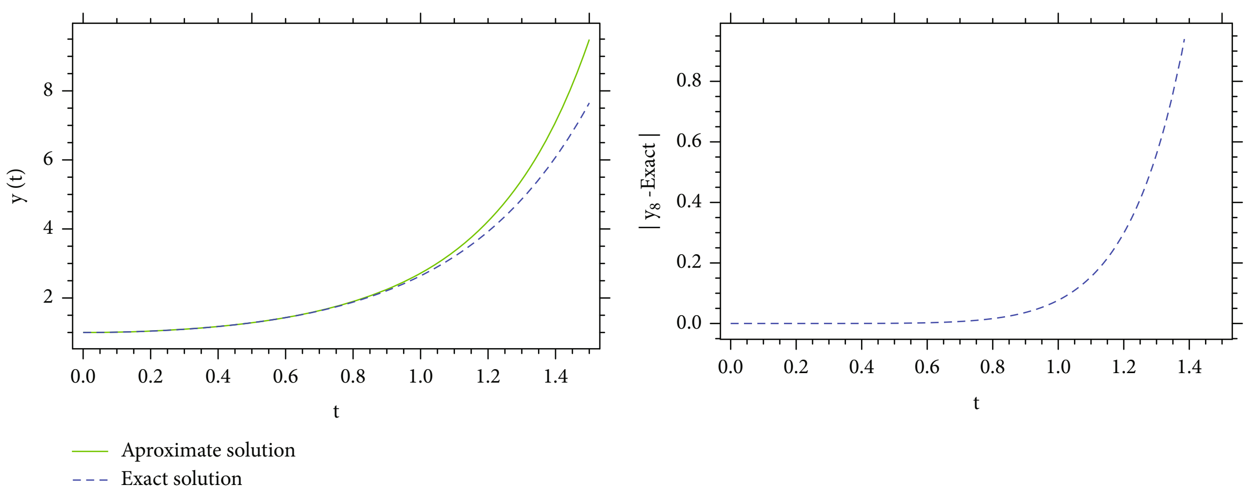
By using the properties of Sumudu transforms listed in Table 2, equation (74) becomes

$$y(t) = 1 + \frac{t^2}{2} - S^{-1}\left[u^2 S\left[2y^2\left(\frac{t}{2}\right)\right]\right]. \tag{75}$$



(a) (b)

Figure 3: (a) 2D solution plots of Example 3 obtained by the present method in comparison with the exact solutions. (b) The absolute errors of Example 3.



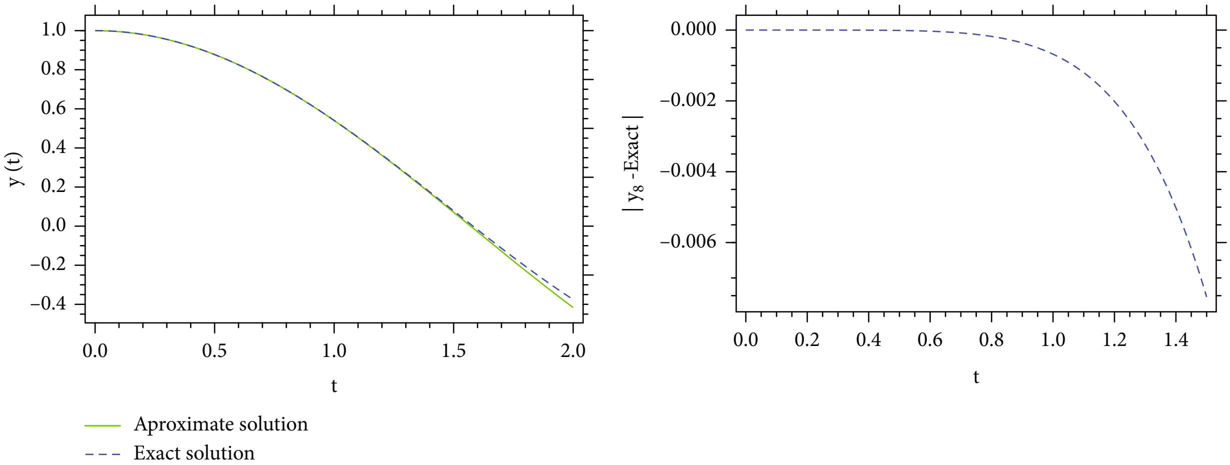
(a) (b)

Figure 4: (a) 2D solution plots of Example 4 obtained by the present method in comparison with the exact solutions. (b) The absolute errors of Example 4.

Now, applying the new iterative method to equation (75) in the understanding of equations (27)–(31), we attain the components of the answer as follows:

$$
\begin{aligned}
y_0(t) &= 1 + \frac{t^2}{2},\\
y_0\left(\frac{t}{2}\right) &= 1 + \frac{t^2}{8},\\
y_1(t) &= -S^{-1}\left[u^2 S\left[2{y_0}^2\left(\frac{t}{2}\right)\right]\right],\\
y_1(t) &= -S^{-1}\left[2u^2 S\left[\left(1+\frac{t^2}{8}\right)^2\right]\right] = -S^{-1}\left[2u^2 + u^4 + \frac{3u^6}{4}\right]\\
&= -\left[t^2 + \frac{t^4}{24} + \frac{t^6}{960}\right] = -t^2 - \frac{t^4}{24} - \frac{t^6}{960},\\
y_1\left(\frac{t}{2}\right) &= -\frac{t^2}{4} - \frac{t^4}{384} - \frac{t^6}{23040},\\
y_2(t) &= N\left[y_0\left(\frac{t}{2}\right) + y_1\left(\frac{t}{2}\right)\right] - N\left[y_0\left(\frac{t}{2}\right)\right]\\
&= -S^{-1}\left[2u^2 S\left[1 - \frac{t^2}{8} - \frac{t^4}{384} - \frac{t^6}{23040}\right]^2 - \left(1+\frac{t^2}{8}\right)^2\right]\\
&= \frac{t^4}{12} + \frac{t^6}{2880} - \frac{6.5t^8}{645120} - \frac{1.3t^{10}}{6635520} + \frac{t^{12}}{583925760}\\
&\quad - \frac{t^{14}}{96613171200}.
\end{aligned} \tag{76}
$$

Similarly, we can obtain the continuing values by consuming the recurrence relation



(a) (b)

Figure 5: (a) 2D solution plots of Example 5 obtained by the present method in comparison with the exact solutions. (b) The absolute errors of Example 4.

Table 3: Eight-term approximate solution by SIM and comparison with the exact solution of Example 1 for different values of time variable t and the absolute error $E = |y_8 - \text{exact}|$.

t	Exact	SIM (y_8)	Error of SIM $\lvert y_8 - exact \rvert$
0.01	0.0099998333341666646825424382691	0.0099998333341666706350847152682 1	$5.9525422769991129423538747960 3 \times 10^{-18}$
0.02	0.019998666693333079366490294692 97	0.019998666693333841257738137566 05	$7.6189124784287308037906972682347 \times 10^{-16}$
0.03	0.029995500202495660768526341926 26	0.029995500202508678106145541852 68	$1.3017337619199926418381976803817 \times 10^{-14}$
0.04	0.039989334186634159452546811715 91	0.039989334186731676342943830011 84	$9.7516890397018295927234489854206 \times 10^{-14}$
0.05	0.049979169270678328794865000845 49	0.049979169271143307005665277342 85	$4.6497821080027649735696632726 69 \times 10^{-13}$
0.06	0.059964006479444599199091137856 98	0.059964006481110618923569582480 79	$1.6660197244784446238102384892945 \times 10^{-12}$
0.07	0.069942847337532763976547306807 89	0.069942847342433740565874528446 27	$4.9009765893272216383775880181497 \times 10^{-12}$
0.08	0.0799146939691726873068763473145	0.079914693981652192127219679558 11	$1.2479504820343332243614165116637 \times 10^{-11}$
0.09	0.089878549198011049691253982607 14	0.089878549226470878124907505214 04	$2.8459828433653522606907291960894 \times 10^{-11}$
0.1	0.099833416646828152306814198410 62	0.099833416706325569403007549568 58	$5.9497417096193351157956953086635 \times 10^{-11}$

Table 4: Eight-term approximate solution by SIM and comparison with the exact solution of Example 2 for different values of time variable t and the absolute error $E = |y_8 - \text{exact}|$.

t	Exact	SIM (y_8)	Error of SIM $\lvert y_8 - \text{exact} \rvert$
0.01	0.980198673306755302220814104225 31	0.980198673298369396053792263885 44	$7.6567061670218403398688662997124 \times 10^{-12}$
0.02	0.960789439152323209439210691323 25	0.960789438882282847002149470899 47	$2.7004036243706122042377588602797 \times 10^{-10}$
0.03	0.941764533584248709537152783271 15	0.941764531520718065013385445731 03	$2.0635306445237673375401197060947 \times 10^{-9}$
0.04	0.923116346386635782910759849572 39	0.923116337636242173234567901234 57	$8.7503936096761919483378188100436 \times 10^{-9}$
0.05	0.904837418035959573164249059446 44	0.904837391164114281298622252449 16	$2.6871845291865626806997276621195 \times 10^{-8}$
0.06	0.886920436717157515527565228769 84	0.886920369431849629875883341256 06	6.7285307885651681887513777352279 × 10 − 8
0.07	0.869358235398805819663084416171 18	0.869358089057390054619381751543 21	$1.463414157650437026646279728325 2 \times 10^{-7}$
0.08	0.852143788966211338456346981468 56	0.852143501871200263449735449735 45	$2.870950110750066115317331126597 7 \times 10^{-7}$
0.09	0.835270211411272021312384974018 78	0.835269690810494082654885428292 41	$5.206007779386574995457263662364 4 \times 10^{-7}$
0.10	0.818730753077981858669935508619 04	0.818729865933817976156029541446 21	$8.871441638825139059671728294243 6 \times 10^{-7}$

Table 5: Eight-term approximate solution by SIM and comparison with the exact solution of Example 3 for different values of time variable t and the absolute error $E = |y_8 - \text{exact}|$.

t	Exact	SIM (y_8)	Error of SIM $|y_8 - \text{exact}|$
0.01	1.0100501670841680575421654569029	1.0100501670841679703001456296863	$8.72407091600392600 33807362201524 \times 10^{-17}$
0.02	1.0202013400267558101601439204832	1.020201340026750198760306630291	$5.61139983729019215 14353035089912 \times 10^{-15}$
0.03	1.0304545339535168556124399538312	1.0304545339534526177307327335903	$6.42378817072202408 98132905025143 \times 10^{-14}$
0.04	1.0408107741923882267570447579169	1.0408107741920254859120141093474	$3.627408450306485694 5474408297705 \times 10^{-13}$
0.05	1.0512710963760240396975176363356	1.0512710963746333515431199754987	$1.390688154397660836 9452201748213 \times 10^{-12}$
0.06	1.0618365465453596222246848771684	1.0618365465411862143015722112639	$4.173407923112665904 4723284282604 \times 10^{-12}$
0.07	1.0725081812542164790531039498891	1.0725081812436398915729504243827	$1.057658748015352550 6414605574959 \times 10^{-11}$
0.08	1.0832870676749585544359877586749	1.083287067651245748758708994709	$2.371280567722904968 0188500198714 \times 10^{-11}$
0.09	1.0941742837052103578728976235449	1.0941742836569532236390447278704	$4.825713423385289567 448601184652 \times 10^{-11}$
0.10	1.1051709180756476248117078264902	1.105170917984387381148296268739	$9.126024366341155775 1246668224547 \times 10^{-11}$

Table 6: Eight-term approximate solution by SIM and comparison with the exact solution of Example 4 for different values of time variable t and the absolute error $E = |y_8 - \text{exact}|$.

t	Exact	SIM (y_8)	Error of SIM $|y_8 - \text{exact}|$
0.01	1.0001000050001666708334166680556	1.00010000500012500156250078125	$2.6046270915886805575397 073415454 \times 10^{-14}$
0.02	1.0004000800106677334186723558807	1.000400080008000400008	$2.6673334106723558806511 753256021 \times 10^{-12}$
0.03	1.0009004051215273424214882074109	1.0009004050911352520238203125	$3.0392090397667894910855 90924096 \times 10^{-11}$
0.04	1.0016012806829398207379736827213	1.001601280512102408192	$1.7083741254597368272128 248212481 \times 10^{-10}$
0.05	1.0025031276057950849746220074108	1.0025031269537354278564453125	$6.5205965711817569491078 540400885 \times 10^{-10}$
0.06	1.0036064877830034418728643444857	1.00360648583462487239203125	$1.9483785694808330944857 236586806 \times 10^{-9}$
0.07	1.0049120246322102296748917218405	1.0049120197151347084	$4.9170755212748917218404 870487403 \times 10^{-9}$
0.08	1.0064205237606613073497170430935	1.006420512794222788608	$1.0966438518741717043093 549369087 \times 10^{-8}$
0.09	1.0081328937531522955843597170773	1.0081328714974127420656328125	$2.2255739553518726904577 318679382 \times 10^{-8}$
0.10	1.0100501670841680575421654569029	1.010050125156328125	$4.1927839932542165456902 860033807 \times 10^{-8}$

Table 7: Eight-term approximate solution by SIM and comparison with the exact solution of Example 5 for different values of time variable t and the absolute error $E = |y_8 - \text{exact}|$.

t	Exact	SIM (y_8)	Error of SIM $\lvert y_8 - \text{exact}\rvert$
0.01	0.99995000041666527778025793375221	0.99995000041666597222322978474719	$6.9444297185099498765700898213503 \times 10^{-16}$
0.02	0.99980000666657777841269559083748	0.99980000666662222196428370810586	$4.4443551588117268384909109084136e \times 10^{-14}$
0.03	0.99955003374898751627215870646661	0.99955003374949374338925373269758	$5.0622711709502623096470099213223 \times 10^{-13}$
0.04	0.99920010666097794031457075812913	0.9992001066638221561884218407565	$2.8442158738510826273615840389857 \times 10^{-12}$
0.05	0.99875026039496624656287081115652	0.9987502604058155784510124137974	$1.0849331888141602640876969231305 \times 10^{-11}$
0.06	0.99820053993520416554766168718284	0.9982005399675983075601052007242	$3.2394142012443513541362920715428 \times 10^{-11}$
0.07	0.99755100025327957462090838993974	0.99755100033496016326004058023897	$8.1680588639132190299235800066344 \times 10^{-11}$
0.08	0.99680170630261938497770677463351	0.99680170648460531599149571403043	$1.8198593101378893939691803779311 \times 10^{-10}$
0.09	0.99595273301199425309283937182514	0.99595273338090037082113388945936	$43.6890611772829451763421912989156 \times 10^{-10}$
0.10	0.99500416527802576609556198780387	0.99500416597212144618056523802984	$6.9409568008500325022596734437406 \times 10^{-10}$

Table 8: The relative error $= |y - y_8|/y$ and the recurrence errors $|y_8 - y_7|$ of the eight-term approximate solution with different values of t for Example 1.

t	Rel.error $= \frac{\lvert y - y_8\rvert}{y}$	The recurrence errors $\lvert y_8 - y_7\rvert$
0.01	$5.9526414871945109656612382108804 \times 10^{-16}$	$1.8771933980659682854623330813807 \times 10^{-42}$
0.02	$3.8097102148157878764411863518049 \times 10^{-14}$	$6.1511873267825648778029730410683 \times 10^{-38}$
0.03	$4.3397634749617773252154115623977 \times 10^{-13}$	$2.6935673489862558793048469387755 \times 10^{-35}$
0.04	$2.4385724939029334238556007347317 \times 10^{-12}$	$2.0156210632401108591584782060973 \times 10^{-33}$
0.05	$9.3034401648822304645892485008952 \times 10^{-12}$	$5.7287396181212411055369051555808 \times 10^{-32}$
0.06	$2.7783662605158795458009676311753 \times 10^{-11}$	$8.8262814891581632653061224489796 \times 10^{-31}$
0.07	$7.0071162039999724917837155730614 \times 10^{-11}$	$8.9120911233770994492519048996914 \times 10^{-30}$
0.08	$1.5616032797619591090961240307272 \times 10^{-10}$	$6.6047871000251952632905013857395 \times 10^{-29}$
0.09	$3.166476171188944712375521506634 \times 10^{-10}$	$3.8649747388840329890348473373724 \times 10^{-28}$
0.1	$5.9596695269552980446935638307384 \times 10^{-10}$	$1.8771933980659682854623330813807 \times 10^{-27}$

$$y_{n+1}(t) = S^{-1}\left[u^2 S\left(\sum_{n=1}^{\infty}\left\{N\left(\sum_{n=0}^{k} {y_n}^2\left(\frac{t}{2}\right)\right) - N\left(\sum_{n=0}^{k-1} {y_n}^2\left(\frac{t}{2}\right)\right)\right\}\right)\right], \quad n \geq 1. \tag{77}$$

Now, in view of (28), the solution of Example 5 is

$$\begin{aligned} y(t) &= y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots \\ &= 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{1440} + \frac{6.5t^8}{645120} - \frac{1.3t^{10}}{6635520} \\ &\quad + \frac{t^{12}}{583925760} - \frac{t^{14}}{96613171200} + . \cdots \end{aligned} \tag{78}$$

6. Discussion

A graph is an essential tool for illustrating the phenomenon physical structures in the context of practical applications. This section has covered the acquired solution that has been shown with figures and tables. The comparison of the eight-term approximate solution using the current technique with the precise answers for illustrative instances $1-5$ at various values of the time variable t is shown in Figures 1(a), 2(a), 3(a), 4(a), and 5(a). These figures demonstrate that the present approach SIM is an effective mathematical instrument. The solution of plots using the present method and the precise solution virtually fit one over the other. Figures 1(b), 2(b), 3(b), 4(b), and 5(b) present the 2D graphs of absolute errors in the interval $t \in (0, 1)$ and over the eight-term approximate and accurate solutions of instances 1, 2, 3, 4, and 5, respectively. There are error functions accessible to separate the scheme's competency and exactness. We provided the absolute, relative, and recurrence error functions to describe the precision and competence of SIM.

The comparisons of eight-term approximation solutions and the exact solutions for instances $1-5$ are shown in Tables 3–7 together with their corresponding absolute errors for various values of the time variable t. The tables show that the current method's answer provides a very excellent estimate with very little error in comparison to the precise solution.

Table 9: Relative error = $|y - y_8|/y$ and the recurrence errors $|y_8 - y_7|$ of the eight-term approximate solution with different values of t for Example 2.

t	Rel. error = $\|y - y_8\|/y$	The recurrence errors $\|y_8 - y_7\|$
0.01	$8.5553127089343170374270459421746 \times 10^{-12}$	$3.7676022376543209876543209876543 \times 10^{-25}$
0.02	$2.8106091869131080159675495546539 \times 10^{-10}$	$3.858024691358024691358024691358 \times 10^{-22}$
0.03	$2.1911322532716372173477594118403 \times 10^{-9}$	$2.2247314453125 \times 10^{-20}$
0.04	$9.4791882344278178169768399760563 \times 10^{-9}$	$3.9506172839506172839506172839506 \times 10^{-19}$
0.05	$2.9697981931597903981571901091965 \times 10^{-8}$	$3.6792990602092978395061728395062 \times 10^{-18}$
0.06	$8.5536390478748811426675600590532 \times 10^{-8}$	$2.2781251507040895061728395061728 \times 10^{-17}$
0.07	$1.6833269624220174754962744684297 \times 10^{-7}$	$1.0642541956018518518518518518519 \times 10^{-16}$
0.08	$3.3690911650403441436132921473112 \times 10^{-7}$	$4.0454320987654320987654320987654 \times 10^{-16}$
0.09	$6.2327229060288257396201891734213 \times 10^{-7}$	$1.313681671142578125 \times 10^{-15}$
0.1	$1.0835603286518002061147680805238 \times 10^{-6}$	$3.7676022376543209876543209876543 \times 10^{-15}$

Table 10: Relative error = $|y - y_8|/y$ and the recurrence errors $|y_8 - y_7|$ of the eight-term approximate solution with different values of t for Example 3.

t	Rel. error = $\|y - y_8\|/y$	The recurrence errors $\|y_8 - y_7\|$
0.01	$8.6373947225877417868639773271375 \times 10^{-17}$	$1.8838011188271604938271604938272 \times 10^{-28}$
0.02	$5.5002866759055889633247305984701 \times 10^{-15}$	$1.929012345679012345679012345679 \times 10^{-25}$
0.03	$6.2339365387389300013823686028123 \times 10^{-14}$	$1.11236572265625 \times 10^{-23}$
0.04	$3.4851757305463662660581726127457 \times 10^{-13}$	$1.9753086419753086419753086419753 \times 10^{-22}$
0.05	$1.3228634927676470260161527926961 \times 10^{-12}$	$1.8396495301046489197530864197531 \times 10^{-21}$
0.06	$3.9303675661670079053871690963782 \times 10^{-12}$	$1.1390625753520447530864197530864 \times 10^{-20}$
0.07	$9.8615448021897730482326991451309 \times 10^{-12}$	$5.3212709780092592592592592592593 \times 10^{-20}$
0.08	$2.1889678539385847003643462470927 \times 10^{-11}$	$2.0227160493827160493827160493827 \times 10^{-19}$
0.09	$4.4103699888137939075348720380715 \times 10^{-11}$	$6.568408355712890625 \times 10^{-19}$
0.1	$8.2575683245733854395955338928169 \times 10^{-11}$	$1.8838011188271604938271604938272 \times 10^{-18}$

Table 11: Relative error = $|y - y_8|/y$ and the recurrence errors $|y_8 - y_7|$ of the eight-term approximate solution with different values of t for Example 4.

t	Rel. error = $\|y - y_8\|/y$	The recurrence errors $\|y_8 - y_7\|$
0.01	$4.1665104197134626794006536633891 \times 10^{-14}$	0.0000000000000000000000078125
0.02	$2.6662666906663110806837260993711 \times 10^{-12}$	0.00000000000000000008
0.03	$3.0364749821414796571902777002511 \times 10^{-11}$	0.0000000000000000004613203125
0.04	$1.7056429124120981178930665544512 \times 10^{-10}$	0.000000000000000008192
0.05	$6.5043154396479848831880033593507 \times 10^{-10}$	0.0000000000000000762939453125
0.06	$1.9413770169868662957457642073733 \times 10^{-9}$	0.00000000000000047239203125
0.07	$4.8930407844154338466912464932543 \times 10^{-9}$	0.0000000000000022068375
0.08	$1.0896477426517253442706479538359 \times 10^{-8}$	0.000000000000008388608
0.09	$2.2076196195387892241157337954729 \times 10^{-8}$	0.0000000000000272405031328125
0.1	$4.151065095467510040888598822550 \times 10^{-8}$	0.000000000000078125

Table 12: Relative error = $|y - y_8|/y$ and the recurrence errors $|y_8 - y_7|$ of the eight-term approximate solution with different values of t for Example 5.

t	Rel. error = $\lvert y - y_8\rvert/y$	The recurrence errors $\lvert y_8 - y_7\rvert$
0.01	$6.9447769544640255824254343490 2116 \times 10^{-14}$	$1.0350555597951431284764618097951e \times 10^{-39}$
0.02	$4.4452441780127633339084847822871 \times 10^{-14}$	$1.6958350291683625016958350291684 \times 10^{-35}$
0.03	$5.0645500475482233460651718232111 \times 10^{-13}$	$4.9506386557778159340659340659341 \times 10^{-33}$
0.04	$2.8464927644529430252097722509013 \times 10^{-12}$	$2.7784561117894451227784561117894 \times 10^{-31}$
0.05	$1.0862907694112761244414947862237 \times 10^{-11}$	$6.3174777819527778837674671008004 \times 10^{-30}$
0.06	$3.2452539060484081575312681432379 \times 10^{-11}$	$8.1111263736263736263736263736264 \times 10^{-29}$
0.07	$8.1881115470179846647898069536872 \times 10^{-11}$	$7.0199856233370383354107312440646 \times 10^{-28}$
0.08	$1.8256984299196189958689336418384 \times 10^{-10}$	$4.5522224935558268891602224935558 \times 10^{-27}$
0.09	$3.7040524665526650385618364415906 \times 10^{-10}$	$2.3678751220786964500343406593407 \times 10^{-26}$
0.1	$6.9758067785681866753931281825812 \times 10^{-10}$	$1.0350555597951431284764618097951 \times 10^{-25}$

Moreover, the proposed approach gives a small error neighboring $t = 0$, but the error increases as $|t|$ grows. This is to means that a greater approximation can be achieved for small values of time t. Additionally, the recurrence and relative errors of the SIM approximation solution for instances $1 - 5$ are generated to verify the correctness of our technique, as shown in Tables 8–12. It is clear from the findings that the current approach is a useful and efficient solution for solving specific classes of nonlinear DDEs with a minimum of computations and iterations.

7. Conclusion

This study introduced SIM, a hybridization of the Sumudu transform method with the new iterative method. First, the linear component of DDEs is solved using the Sumudu transform method. To simplify the complexity of the novel term from the nonlinear term, a posttreatment new iterative method is employed as illustrated in Section 3. We provide the fundamental definitions and terminology for the DDEs, the new iterative approach, and the Sumudu transform method. The validity and consistency of SIM have been verified with the help of five significant problems. The absolute, recurrence, and relative errors of all considered examples are interpreted graphically and numerically, for different values of the time variable t. From the demonstrative examples, the results reveal that the current technique SIM generates a decent approximation that is extremely close to the precise answers with a low amount of error. Therefore, SIM is quite valuable, as it allows us to improve accuracy and efficiency and provides a mathematical tool for nonlinear DDEs. Finally, we hope that this work is a step toward examining this approach to tackling some exciting problems in various areas of science and engineering, given the ongoing use of nonlinear DDEs as models in several different disciplines.

Data Availability

No data were used to support the study.

Conflicts of Interest

There are no conflicts of interest declared by the authors.

Authors' Contributions

The authors all participated equally and approved the submitted version of the paper.

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