Hindawi
Advances in Mathematical Physics
Volume 2022, Article ID 9698299, 26 pages
https://doi.org/10.1155/2022/9698299

Research Article

Unified Approach to Fractional Calculus Images Involving the Pathway Transform of Extended k-Gamma Function and Applications

Asifa Tassaddiq [1] **and Kwara Nantomah** [2]

[1] *Department of Basic Sciences and Humanities, College of Computer and Information Sciences Majmaah University, Al Majmaah 11952, Saudi Arabia*
[2] *Department of Mathematics, School of Mathematical Sciences, C.K. Tedam University of Technology and Applied Sciences, P. O. Box 24, Navrongo, Upper East Region, Ghana*

Correspondence should be addressed to Asifa Tassaddiq; a.tassaddiq@mu.edu.sa
and Kwara Nantomah; knantomah@cktutas.edu.gh

Received 17 February 2022; Revised 17 May 2022; Accepted 11 June 2022; Published 22 July 2022

Academic Editor: Zine El Abiddine Fellah

A path way model is concerned with the rules of swapping among different classes of functions. Such model is significant to fit a parametric class of distributions for new data. Based on a such model, the path way or P_δ transform is of binomial type and contains a family of different transforms. Taking inspiration from these facts, present research is concerned with the computation of new fractional calculus images involving the extended k-gamma function. The non-integer kinetic equations containing the extended k-gamma function is solved by using pathway transform as well as validated with the earlier obtained results. P_δ transform of Dirac delta function is obtained which proved useful to achieve the purpose. As customary, the results for the frequently used Laplace transform can be recovered by taking $\delta \longrightarrow 1$ in the definition of P_δ transform. Important new identities involving the Fox-Wright function are obtained and used to simplify the results. It is remarkable that the several new and novel results involving the classical gamma function became possible by using this approach.

1. Introduction

Research and studies reveal that a large number of integral transforms are found in the literature. Each one of them is suitable for different problems despite a simple mathematical relation existing between them. This role is vital to understanding and applying the modern and classical theories. More recently, fractional type integral transform called P_δ transform or pathway transform is introduced in [1] based on pathway model [2, 3]. This model is significant to study important forms of statistics [4, 5] due to their applications in astrophysics [6–8] and applied analysis [9–11]. By varying the pathway parameter, one can obtain three useful forms. For $a>0, \delta>0, \gamma>0, \eta>0$, and the normalizing constants c_1, c_2, c_3; the pathway model is defined as follows:

$$f(x)=\begin{cases} c_1|x|^\gamma\left[1-a(1-\alpha)|x|^\delta\right]^{\eta/(1-\alpha)}, 1-a(1-\alpha)|x|^\delta>0, \alpha<1 \\ c_2|x|^\gamma\left[1-a(1-\alpha)|x|^\delta\right]^{-\eta/(1-\alpha)}, -\infty<x<\infty, \alpha>1 \\ c_3|x|^\gamma e^{-a\eta|x|^\delta}, -\infty<x<\infty, \alpha\longrightarrow 1 \end{cases}. \tag{1}$$

For a complex valued integrable function $f(t), P_\delta$ transform is defined as follows [1]:

$$P_\delta[f(t);s]=F(s)=\int_0^\infty[1+(\delta-1)s]^{(-t)/(\delta-1)}f(t)dt, \quad \delta>1; s\in\mathbb{C}, t\in\mathbb{R}, \tag{2}$$

which is convergent for $\mathfrak{R}(\ln[1+(\delta-1)s]/\delta-1)>0$. Involved variable t is transformed from $\ln[1+(\delta-1)s]^{-t/\delta-1}$ to e^{-st} similar to as the paths are transformed from binomial to the exponential under pathway transformations [1].

Agarwal et al. [12] recently used such transforms to solve the non-integer order differential and integral equations. Srivastava et al. [13, 14] used this transform to find certain results involving different special functions. For more details, see [1–15] and references therein. Some properties of the pathway transform are given in [1, 12]. For example, the following result is detailed in [1]:

$$P_\delta[z^{\chi-1};s]=\Gamma(\chi)\left\{\frac{\delta-1}{\ln[1+(\delta-1)s]}\right\}^{\chi},\quad \frac{t^{\chi-1}}{\Gamma(\chi)}=P_\delta^{-1}\left\{\frac{\delta-1}{\ln[1+(\delta-1)s]}\right\}^{\chi}. \tag{3}$$

The pathway or P_δ transform of Riemann–Liouville fractional derivatives of function $f(t)$ of order ν is computed by [1]:

$$P_\delta\{I_{0+}^{\nu}\Omega(t);s\}=\left(\frac{\delta-1}{\ln[1+(\delta-1)s]}\right)^{\nu}\Omega(s). \tag{4}$$

The pathway or P_δ transform of Caputo derivatives of function $f(t)$ of non-integer order ν and $n-1<\nu\le n$ is given by [12]:

$$P_\delta[{}_0^cD_t^\nu f(t);s]=\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu}F(s)-\sum_{k=0}^{n-1}\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{\nu-k-1}f^{(k)}(0). \tag{5}$$

If $F(s)=P_\delta[f(t);s]$ and $G(s)=P_\delta[g(t);s]$, then the convolution theorem is stated in [1] as follows:

$$F(s)G(s)=P_\delta[f(t);s]P_\delta[g(t);s]. \tag{6}$$

The P_δ transform (2) diminishes to Laplace transform (Sneddon [16]) in the limiting case as $\delta\longrightarrow 1$:

$$L[f(t);s]=\int_0^\infty e^{-st}f(t)dt;\mathfrak{R}(s)>0, \tag{7}$$

which is a classical tool to solve nontrivial problems [17–19] of diverse nature. It is remarkable that the following relation between two transforms exist [13–15]:

$$L[f(t);s]=P_\delta\left[f(t);\frac{e^{(s-1)\delta}-1}{\delta-1}\right],\quad P_\delta[f(t);s]=L\left[f(t);\frac{\ln[1+(\delta-1)s]}{\delta-1}\right]. \tag{8}$$

Hence, by using (2), (7), and (8), the pathway or P_δ transform of delta function is given by

$$L\left\{\delta^{(r)}(z);\xi\right\}=(\xi)^r \text{ and } P_\alpha\left\{\delta^{(r)}(z);\xi\right\}=\left(\frac{\ln[1+(\delta-1)\xi]}{\delta-1}\right)^{r}. \tag{9}$$

A nontrivial generalization of the classical gamma function is investigated by Chaudhry and Zubair [20]:

$$\Gamma_b(z)=\int_0^\infty t^{z-1}e^{-t-(b/t)}dt,\ (\sigma=\mathfrak{R}(z)>0,b\ge 0), \tag{10}$$

which attracted the attention of many researchers due to its applications, and therefore further extensions were focused on by them [21]. One such generalization is named as the "extended k-gamma function" given by [21]:

$$\Gamma_{b,k}(z)=\int_0^\infty t^{z-1}\exp\left(\frac{-t^k-\left(b^k/t^k\right)}{k}\right)dt,\ (\mathfrak{R}(z)>0,b\ge 0,k>0). \tag{11}$$

When k is unity, it diminishes to (10), and when b is zero, it becomes k-gamma function defined by [22]:

$$\Gamma_k(z)=\int_0^\infty t^{z-1}e^{\left(-t^k\right)/k}dt\quad (k\in\mathbb{R}), \tag{12}$$

and when $k=1$, the *classical* gamma function $\Gamma_1(z)=\Gamma(z)$ is obtained, while for $k=2$, it is significant to investigate the important Gaussian distributions [23, 24]. k-gamma function has diverse applications for example, in physics and chemistry [24, 25]; in non-integer order calculus [26, 27]; and in statistical analysis [28]. The interested reader is directed to [29–33] and associated references therein for a more extensive and exhaustive review of related work. In recent years, Tassaddiq [34] has investigated a distributional representation of the extended k-gamma function in terms of complex delta function as follows:

$$\Gamma_{b,k}(z)=2\pi\sum_{n,r=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}}{n!r!}\delta(z+nk-kr), \tag{13}$$

$$\Gamma_{b,k}(z)=2\pi\sum_{n,r,p=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}(k)^p(n-r)^p}{n!r!p!}\delta^{(p)}(z). \tag{14}$$

Hence, its action on a suitably chosen function over a specific domain is easily obtainable by means of the standard techniques applied to delta function, and it is found that [34]

$$L(\Gamma_{b,k}(z);\xi) = 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-1/k)^{n+r}(b)^{kr}(k)^{p}(n-r)^{p}}{n!r!p!}\xi^{p} = 2\pi exp\left(-\frac{e^{k\xi}}{k} - \frac{b^{k}e^{-k\xi}}{k}\right). \quad (15)$$

For further information on such representations concerning other special functions, and for properties of delta functions, one may refer to [34–45].

1.1. Novelty and Significance of the Research. The gamma function is a basic and widely studied function, but if we google "Laplace transform of gamma function," then we cannot find it in the existing literature [20] (see also https://math.stackexchange.com/questions/2938166/what-is-the-laplace-transform-of-gamma-function).

More recently, it has been computed by Tassaddiq in [34] by using the distributional representation of the extended k-gamma function. Present motivation is to further explore and extend the results of [34] keeping in view their novelty and significance to crack the problems that remained unsolved for many years. For instance, quite a few recent articles by Kiryakova ([46, 47] and references therein) point out that the special functions are either generalized fractional calculus operators of the basic functions (or the generalized fractional calculus operators are defined as the action of a specific special function on a general class of functions and vice versa). This marriage of special functions and fractional calculus is astounding. As a result, several authors have worked to compute the image formulae of a wide range of special functions using important fractional transforms. Such findings are discussed in a review article [46], and [15, 48, 49] contain an interesting debate between the researchers. The authors are inspired by these fractional operators because they connect numerous frequently used fractional operators [46]. As a result, we use the generalized fractional calculus operators to calculate the novel fractional images of the extended k-gamma function. They are known as generalized fractional integrals (multiple E–K operators) in [46, p. 8, Equation (19)]. In addition, the lately common and highly used fractional operators Riemann–Liouville (R-L), Saigo, and Marichev–Saigo–Maeda (M-S-M) are described with reference to such generalized operators. This approach has been used first time in this research and was not possible to use it for the famous gamma function (and its generalizations) by using its known (old) and classical representations.

This study plans as follows: Section 2 contains the necessary preliminaries related to the family of generalized fractional integrals (multiple E–K operators) and involved special functions. Section 3.1 includes fractional images that use the gamma function and its extensions. Section 3.2 goes over fractional derivatives. The following Section 3.3 focuses on the formulation and solution of a non-integer order kinetic equation using the pathway transform. Section 3.4 computes integrals involving the products of a class of special functions. Section 4 includes conclusion with a detailed discussion and comparison of the findings with other studies.

2. Preliminaries

Note: In this research, $\mathfrak{R}$, $\mathbb{C}$, and $\mathbb{R}$ are the symbols to denote the real part of any complex number, complex numbers, and real numbers separately. $\mathbb{R}^{+}$ is a set of positive reals, and $\mathbb{Z}_0^{-}$ is a set of negative integers including 0.

Definition 1. (see [50]). For $\alpha\epsilon\mathbb{C};\mathfrak{R}(\alpha)>0$, Mittag-Leffler function is defined in a series form:

$$E_{\alpha}(z) = \sum_{r=0}^{\infty} \frac{z^{r}}{\Gamma(\alpha r+1)}. \quad (16)$$

$\Gamma(\alpha r+1)$ denotes the familiar gamma function [20], and, when $\alpha=1$, it diminishes to the factorial, and hence Mittag-Leffler function diminishes to the exponential function. Similarly, Mittag-Leffler function of parameters 2 and 3 i-e $(\alpha, \beta, \gamma\epsilon\mathbb{C}, \mathfrak{R}(\alpha)>0)$ is defined as

$$E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^{r}}{\Gamma(\alpha r+\beta)}; E_{\alpha,\beta}^{\gamma}(z) = \sum_{r=0}^{\infty} \frac{(\gamma)_{r}z^{r}}{\Gamma(\alpha r+\beta)}. \quad (17)$$

Definition 2 (see [46, 51, 52]). The Fox-Wright function $({}_{p}\Psi_{q})$ has a series representation

$${}_{p}\Psi_{q}\left[\begin{matrix}(a_i, A_i)\\(b_j, B_j)\end{matrix};z\right] = \sum_{m=0}^{\infty} \frac{\prod_{l=1}^{p}\Gamma(a_i + A_i m)}{\prod_{l=1}^{q}\Gamma((b_j + B_j m)} \frac{z^m}{m!} \cdot \left(a_i\epsilon\mathbb{R}^{+}(i=1,\cdots.p);B_j\epsilon\mathbb{R}^{+}(j=1,\cdots.q); 1+\sum_{i=1}^{q}B_i - \sum_{j=1}^{p}A_j > 0\right). \quad (18)$$

Definition 3 (see [46, 51, 52]). The fox H-function is defined as follows:

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i, A_i)\\(b_j, B_j)\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1, A_1), \cdots, & (a_i, A_i)\\(b_1, B_1), \cdots, & (b_j, B_j)\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{\mathscr{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j\mathfrak{s})\prod_{i=1}^{n}\Gamma(1-a_j - A_j\mathfrak{s})}{\prod_{j=m+1}^{q}\Gamma(1-b_j - B_j\mathfrak{s})\prod_{i=n+1}^{p}\Gamma(a_j + A_j\mathfrak{s})} z^{-\mathfrak{s}} d\mathfrak{s} \quad (1\le m\le q; 0\le n\le p; A_i>0; B_j>0; a_i, b_j\in\mathbb{C}; i=1,\cdots,p; j=1,\cdots,q), \quad (19)$$

where $\mathscr{L}$ denotes a suitable Mellin-Barnes curve to keep secluded the poles of $\{\Gamma(b_j + B_j\mathfrak{s})\}_{j=1}^{m}$ from that of $\{\Gamma(1-a_j - A_j\mathfrak{s})\}_{j=1}^{n}$.

Remark 4 (see [46, 51, 52]). If $A_p = B_q = 1$, then H-function turns into Meijer G-function:

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1),\cdots, & (a_i,A_i)\\ (b_1,B_1),\cdots, & (b_j,B_j)\end{matrix}\right.\right] = \sum_{m=0}^{\infty}\frac{\prod_{j=1}^{m}\Gamma(b_j+B_jm)\prod_{i=1}^{n}\Gamma(1-a_i-A_im)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_jm)\prod_{i=n+1}^{p}\Gamma(a_i+im)}\frac{z^m}{m!},$$

$$\mathrm{H}_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,1),\cdots, & (a_i,1)\\ (b_1,1),\cdots, & (b_j,1)\end{matrix}\right.\right] = \mathrm{G}_{p,q}^{m,n}\left[z\left|\begin{matrix}a_1\cdots, & a_i\\ b_1,\cdots, & b_j\end{matrix}\right|\right], \quad (20)$$

and the Fox-Wright function is related with H-function

$$_p\Psi_q\left[\begin{matrix}(a_i,A_i)\\ (b_j,B_j)\end{matrix};z\right] = H_{p,q+1}^{1,p}\left[-z\left|\begin{matrix}(1-a_1,A_1),\cdots, & (1-a_i,A_i)\\ (0,1),(1-b_1,B_1),\cdots, & ((1-b_j,B_j)\end{matrix}\right.\right], \quad (21)$$

and the hypergeometric functions as

$$_p\Psi_q\left[\begin{matrix}(a_i,1)\\ (b_j,1)\end{matrix};z\right] = G_{p,q+1}^{1,p}\left[-z\left|\begin{matrix}(1-a_1,1),\cdots, & (1-a_i,1)\\ 0,(1-b_1,1),\cdots, & (1-b_j,1)\end{matrix}\right.\right] = {}_p\mathrm{F}_q\left[\begin{matrix}a_i\\ b_j\end{matrix};z\right]\cdot\frac{\Gamma(a_1)\cdots\Gamma(a_i)}{\Gamma(b_1)\cdots\Gamma(b_j)}. \quad (a_i>0;b_j\notin\mathbb{Z}_0^-). \quad (22)$$

Definition 5 (see [46]). The generalized fractional integrals namely (multiple) E–K operators as defined in are

$$I_{(\beta_k),m}^{(\gamma_k),(\nu_k)}f(z) = \begin{cases}\displaystyle\int_0^1 f(z\sigma)\,H_{m,m}^{m,0}\left[\sigma\left|\begin{matrix}\left(\gamma_k+\nu_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right.\right]d\sigma;\sum_k\nu_k>0\\ \displaystyle z^{-1}\int_0^z f(\xi)\,H_{m,m}^{m,0}\left[\frac{\xi}{z}\left|\begin{matrix}\left(\gamma_k+\nu_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right.\right]d\xi;\sum_k\nu_k>0\end{cases}. \quad (23)$$

Here, the order of integration is represented by $\nu_k{}'s$, the additional parameters are represented by $\beta_k{}'s$, and weights are symbolized by $\gamma_k{}'s$. It is important to notice that $\mathrm{I}_{(\beta_k),m}^{(\gamma_k),(\nu_k)}\mathrm{f}(z) = \mathrm{f}(z);\nu_k = 1$. Since the function $H_{m,m}^{m,0}$ cease to exist for $|\sigma|>1$, therefore the upper limit in (23) can be replaced by ∞.

Definition 6. (see [46]). Similarly, the fractional derivative of order $\nu = (\nu_1 \ge 0,\cdots,\nu_m \ge 0)$ in response to (23) are defined as

$$D_{(\beta_k),m}^{(\gamma_k)_1^m,(\nu_k)}(f(z)) := D_\eta I_{(\beta_k),m}^{(\gamma_k+\nu_k),(\eta_k-\nu_k)}f(z) = D_\eta\int_0^1 f(z\sigma)H_{m,m}^{m,0}\left[\sigma\left|\begin{matrix}\left(\gamma_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right.\right]d\sigma, \quad (24)$$

where D_η (polynomial of degree $\eta_1+\cdots+\eta_m$ in variable $z(d/dz)$) is

$$D_\eta = \prod_{r=1}^{m}\prod_{j=1}^{\eta_r}\frac{1}{\beta_r}z\frac{d}{dz}+\gamma_r+j;\eta_k = \begin{cases}[\nu_k]+1;\nu_k\notin\mathbb{Z}\\ \nu_k;\nu_k\in\mathbb{Z}\end{cases}. \quad (25)$$

Furthermore, the conforming non-integer order derivatives of Caputo type are

$$*\mathrm{D}_{(\beta k),\mathrm{m}}^{(\gamma k)_1^{\mathrm{m}},(\nu k)}\mathrm{f}(z) = \mathrm{I}_{(\beta k),\mathrm{m}}^{(\gamma k+\nu k),(\eta_k-\nu_k)}\mathrm{D}_\eta\mathrm{f}(z). \quad (26)$$

Lemma 7 (see [46]). *For* $[-\beta_k(1+\gamma_k)]<p;\nu_k\ge 0;k=1,\cdots,m$, *we have*

$$I_{(\beta k),m}^{(\gamma k),(\nu k)}\{z^p\} = \prod_{i=1}^{m}\frac{\Gamma(\gamma_i+1+p/\beta_i(l))}{\Gamma(\gamma_i+\nu_i+1+(p/\beta_i))}z^p. \quad (27)$$

Definition 8 (see [52–56]). For complex parameters $\gamma_1,\gamma_1{}',\gamma_2,\gamma_2{}',\Re(\nu)>0$, the M-S-M fractional integral operators are defined by

$$\left(\mathrm{I}_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\mathrm{f}\right)(\mathrm{x}) = \frac{\mathrm{x}^{-\gamma_1}}{\Gamma(\nu)}\int_0^{\mathrm{x}}(\mathrm{x}-\mathrm{t})^{\nu-1}\mathrm{t}^{-\gamma_1{}'}\mathrm{F}_3\cdot\left(\gamma_1,\gamma_1{}',\gamma_2,\gamma_2{}',\nu;1-\frac{\mathrm{t}}{\mathrm{x}};1-\frac{\mathrm{x}}{\mathrm{t}}\right)\mathrm{f}(\mathrm{t})\mathrm{dt},$$

$$\left(\mathrm{I}_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\mathrm{f}\right)(\mathrm{x}) = \frac{\mathrm{t}^{-\gamma_1{}'}}{\Gamma(\nu)}\int_{\mathrm{x}}^{\infty}(\mathrm{x}-\mathrm{t})^{\nu-1}\mathrm{t}^{-\gamma_1}\mathrm{F}_3\cdot\left(\gamma_1,\gamma_1{}',\gamma_2,\gamma_2{}',\nu;1-\frac{x}{t};1-\frac{t}{x}\right)\mathrm{f}(\mathrm{t})\mathrm{dt}. \quad (28)$$

where F_3 represents Appel Function (Horn function) which is defined as [52]

$$\mathrm{F}_3\left(\gamma_1,\gamma_1{}',\gamma_2,\gamma_2{}',\nu;\mathrm{u};\mathrm{v}\right) = \sum_{\mathrm{k,l}=0}^{\infty}\frac{(\gamma_1)_k\left(\gamma_1{}'\right)_l(\gamma_2)_k\left(\gamma_2{}'\right)_l}{(\nu)_{l+m}}\frac{u^{\mathrm{k}}}{k!}\frac{v^{\mathrm{l}}}{l!}, \max(|u|,|v|)<1. \quad (29)$$

Lemma 9 (see [52–56]). *Let* $\gamma_1, \gamma_1', \gamma_2, \gamma_2' \in \mathbb{C}, \omega > 0 \wedge \Re(\chi) > \max\{0, \Re(\gamma_1 + \gamma_1' + \gamma_2 - \nu), \Re(\gamma_1' - \gamma_2')\}, \Re(\nu) > 0$, *then*

$$I_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\left(\omega^{\chi-1}\right) = \frac{\Gamma(\chi)\Gamma\left(\chi+\nu-\gamma_1-\gamma_1'-\gamma_2\right)\Gamma\left(\chi+\gamma_2'-\gamma_1'\right)}{\Gamma\left(\chi+\gamma_2'\right)\Gamma\left(\chi+\nu-\gamma_1-\gamma_1'\right)\Gamma\left(\chi+\nu-\gamma_1'-\gamma_2\right)} \cdot \omega^{\nu+\chi-\gamma_1-\gamma_1'-1}. \tag{30}$$

Similarly, let $\gamma_1, \gamma_1', \gamma_2, \gamma_2' \in \mathbb{C}, \omega > 0$, *and if* $\Re(\delta) > 0$, $\Re(\chi) < 1 + \min\{\Re(-\gamma_2), \Re(\gamma_1 + \gamma_1' - \delta), \Re(\gamma_1 + \gamma_2' - \nu)\}$ *, then following image formula holds true:*

$$I_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\left(\omega^{\chi-1}\right) = \frac{\Gamma\left(1-\chi-\nu+\gamma_1+\gamma_1'\right)\Gamma\left(1-\chi+\gamma_1+\gamma_2'-\nu\right)\Gamma(1-\chi-\gamma_1)}{\Gamma(1-\chi)\Gamma\left(1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\nu\right)\Gamma(1-\chi+\gamma_1-\gamma_2)} \cdot \omega^{\nu+\chi-\gamma_1-\gamma_1'-1}. \tag{31}$$

Definition 10. (see [55]). For $\gamma_1, \gamma_2, \nu \in \mathbb{C}$ with $x\,; \Re(\nu) > 0$ by Saigo fractional integral operators are defined by

$$\begin{aligned} I_{0+}^{\gamma_1,\gamma_2,\nu} &= \frac{x^{-\nu-\gamma_1}}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}{}_2F_1\left(\nu+\gamma_2,-\gamma_1\,;\nu\,;1-\frac{t}{x}\right)f(t)dt, \\ I_{-}^{\gamma_1,\gamma_2,\nu}(f(x)) &= \frac{1}{\Gamma(\nu)}\int_x^\infty (t-x)^{\nu-1}t^{-\nu-\gamma_1}{}_2F_1\left(\nu+\gamma_2,-\gamma_1\,;\nu\,;1-\frac{x}{t}\right)f(t)dt, \end{aligned} \tag{32}$$

where ${}_2F_1$ represents the Gauss hypergeometric function given by (see [56])

$$\begin{aligned} {}_2F_1(\gamma_1,\gamma_2,\gamma_3\,;u) &= \sum_{k=0}^{\infty}\frac{(\gamma_1)_k(\gamma_2)_k}{(\gamma_3)_k}\frac{u^k}{k!}, |u|<1\,;|u| \\ &= 1(u\neq 1), \Re(\gamma_3-\gamma_1-\gamma_2) > 0. \end{aligned} \tag{33}$$

Lemma 11 (see [53–56]). *For* $\gamma_1, \gamma_2, \nu \in \mathbb{C}\,; \Re(\nu) > 0, \Re(\chi) > \max[0, \Re(\gamma_1 - \gamma_2)]$, *we have*

$$I_{0+}^{\gamma_1,\gamma_2,\nu}\left(\omega^{\chi-1}\right) = \frac{\Gamma(\chi)\Gamma(\chi+\gamma_2-\gamma_1)}{\Gamma(\chi-\gamma_2)\Gamma(\chi+\nu+\gamma_2)}\omega^{\chi-\gamma_1-1}, \tag{34}$$

and for $\gamma_1, \gamma_2, \nu \in \mathbb{C} \wedge \Re(\nu) > 0 \wedge \Re(\chi) < 1 + \min[\Re(\gamma_1), \Re(\gamma_2)]$:

$$I_{-}^{\gamma_1,\gamma_2,\nu}\left(\omega^{\chi-1}\right) = \frac{\Gamma(\gamma_1-\chi+1)\Gamma(\gamma_2-\chi+1)}{\Gamma(1-\chi)\Gamma(\gamma_1+\gamma_2+\nu-\chi+1)}\omega^{\chi-\gamma_1-1}. \tag{35}$$

Definition 12 (see [53–56]). For complex $\gamma, \nu \in \mathbb{C}, \Re(\nu) > 0$, Erdélyi–Kober integrals are defined by

$$\begin{aligned} I_{0+}^{0,\gamma,\nu}(f(x)) &= \left(I_{0+}^{\gamma,\nu}f\right)(x) = \frac{x^{-\nu-\gamma}}{\Gamma(\chi)}\int_0^x (x-t)^{\nu-1}t^{\gamma}f(t)dt \quad (x>0), \\ I_{0-}^{0,\gamma,\nu}(f(x)) &= \left(I_{0-}^{\gamma,\nu}f\right)(x) = \frac{x^{\gamma}}{\Gamma(\chi)}\int_x^\infty (t-x)^{\nu-1}t^{-\nu-\gamma}f(t)dt \quad (x>0). \end{aligned} \tag{36}$$

Definition 13 (see [57, 58]). For $\nu \in \mathbb{C} \wedge \Re(\nu) > 0$, the R-L fractional integrals are defined as

$$\begin{aligned} I_{0+}^{\nu}(f(x)) &= \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)dt \quad (x>0), \\ I_{-}^{\nu}(f(x)) &= \frac{1}{\Gamma(\nu)}\int_x^\infty (x-t)^{\nu-1}f(t)dt \quad (x>0), \end{aligned} \tag{37}$$

respectively. These are also related to Weyl transform [57, 58].

Remark 14. All of the above fractional operators are related and can be obtained as special cases of (17) by varying and specifying different parameter values. This is described below in the following Table 1.

3. Main Results

3.1. New Fractional Image Formulae Containing the Extended k-Gamma Function

Lemma 15. *Assuming* $b, k > 0$, *prove that the subsequent relation for the Fox-Wright function holds true:*

$$\begin{aligned} &\sum_{n,r=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}}{n!r!}{}_0\Psi_0\left[\begin{array}{c|c} - & \\ - & \end{array} k(n-r)\xi\right] \\ &= {}_0\Psi_0\left[\begin{array}{c|c} - & \\ - & \end{array} -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right]. \end{aligned} \tag{38}$$

Proof. Let us consider (14), then

$$L(\Gamma_{b,k}(z)\,;\xi) = 2\pi\sum_{n,r=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}}{n!r!}{}_0\Psi_0\left[\begin{array}{c|c} - & \\ - & \end{array} k(n-r)\xi\right], \tag{39}$$

$$\begin{aligned} L(\Gamma_{b,k}(z)\,;\xi) &= 2\pi\exp\left(-\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right) \\ &= {}_0\Psi_0\left[\begin{array}{c|c} - & \\ - & \end{array} -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right]. \end{aligned} \tag{40}$$

Hence, from both of the above Equations (39)–(40), the required result is proved. □

Table 1: Important cases of (multiple) E–K operators.

Cases of (23)	Relation between kernels of the above fractional operators [46]
$(m=3\,;\beta_1=\beta_2=\beta_3=\beta=1)$ Marichev–Saigo–Maeda (M-S-M)	$H_{3,3}^{3,0}\left(\frac{t}{x}\right)=G_{3,3}^{3,0}\left[\frac{t}{x}\left\vert\begin{matrix}\gamma_1'+\gamma_2',\boldsymbol{\nu}-\gamma_1,\boldsymbol{\nu}-\gamma_2\\ \gamma_1',\gamma_2',\boldsymbol{\nu}-\gamma_1-\gamma_2\end{matrix}\right.\right]=\frac{x^{-\gamma_1}}{\Gamma(\boldsymbol{\nu})}(x-t)^{\delta-1}t^{-\gamma_1'}F_3\left(\gamma_1,\gamma_1',\gamma_2,\gamma_2',\boldsymbol{\nu};1-\frac{t}{x};1-\frac{x}{t}\right)$
$\left(m=2\,;\beta_1=\beta_2=\beta>0\,;\sigma=\frac{t}{x}\wedge\sigma=\frac{x}{t}\right)$ Saigo	$H_{2,2}^{2,0}\left[\sigma\left\vert\begin{matrix}\left(\gamma_1+\boldsymbol{\nu}_1+1-\frac{1}{\beta},\frac{1}{\beta}\right),\left(\gamma_2+\boldsymbol{\nu}_2+1-\frac{1}{\beta},\frac{1}{\beta}\right)\\ \left(\gamma_1+1-\frac{1}{\beta},\frac{1}{\beta}\right),\left(\gamma_2+1-\frac{1}{\beta},\frac{1}{\beta}\right)\end{matrix}\right.\right]=G_{2,2}^{2,0}\left[\sigma^{\beta}\left\vert\begin{matrix}\gamma_1+\boldsymbol{\nu}_1,\gamma_2+\boldsymbol{\nu}_2\\ \gamma_1,\gamma_2\end{matrix}\right.\right]=\beta\frac{\sigma^{\beta\gamma_2}\left(1-\sigma^{\beta}\right)^{\boldsymbol{\nu}_1+\boldsymbol{\nu}_2-1}}{\Gamma(\boldsymbol{\nu}_1+\boldsymbol{\nu}_2)}{}_2F_1\left(\gamma_2+\boldsymbol{\nu}_2-\gamma_1,\boldsymbol{\nu}_1;\boldsymbol{\nu}_1+\boldsymbol{\nu}_2;1-\sigma^{\beta}\right)$
$(m=1)$ Erdélyi–Kober (E-K)	$H_{1,0}^{1,1}\left[\sigma\left\vert\begin{matrix}\left(\gamma+\boldsymbol{\nu},\frac{1}{\beta}\right)\\ \left(\gamma,\frac{1}{\beta}\right)\end{matrix}\right.\right]=\beta\sigma^{\beta-1}G_{1,0}^{1,1}\left[\sigma^{\beta}\left\vert\begin{matrix}\gamma+\boldsymbol{\nu}\\ \gamma\end{matrix}\right.\right]=\beta\frac{\sigma^{\beta\gamma+\beta-1}\left(1-\sigma^{\beta}\right)^{\boldsymbol{\nu}-1}}{\Gamma(\boldsymbol{\nu})}$
$\left(m=1\,;\beta=1\,;\sigma=\frac{t}{x}\wedge\sigma=\frac{x}{t}\right)$ Riemann–Liouville (R-L)	$H_{1,0}^{1,1}\left[\sigma\left\vert\begin{matrix}(\gamma+\boldsymbol{\nu},1)\\ (\gamma,1)\end{matrix}\right.\right]=G_{1,0}^{1,1}\left[\frac{t}{x}\left\vert\begin{matrix}\gamma+\boldsymbol{\nu}\\ \gamma\end{matrix}\right.\right]=\frac{(x-t)^{\boldsymbol{\nu}-1}t^{\gamma}}{\Gamma(\boldsymbol{\nu})}$

Remark 16. It is to be remarked that a general result is obvious from (38) as follows:

$$\sum_{n,r=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}}{n!r!}{}_p\Psi_q\left[\begin{matrix}(a_i,A_i)\\(b_j,B_j)\end{matrix}\middle|k(n-r)\xi\right]={}_p\Psi_q\left[\begin{matrix}(a_i,A_i)\\(b_j,B_j)\end{matrix}\middle|-\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right].\tag{41}$$

Similar results will also hold for Mittag-Leffler function and other related functions.

Remark 17. A similar result can be proved by using the path way transform. It can be deduced from (38) by using the following replacement on LHS:

$$\xi\longrightarrow\frac{\ln\left[1+(\delta-1)\xi\right]}{\delta-1}.\tag{42}$$

Henceforth,

$$\begin{aligned}P_\delta(\Gamma_{b,k}(z);\xi)&=2\pi\sum_{n,r,p=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}(k)^p(n-r)^p}{n!r!p!}\left(\frac{\ln\left[1+(\delta-1)\xi\right]}{\delta-1}\right)^p\\&=2\pi exp\left(-\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right)\\&=\mathbf{2\pi_0\Psi_0}\left[\begin{matrix}-\\-\end{matrix}\middle|-\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right].\end{aligned}\tag{43}$$

Or equivalently, in equation (43), 2\pi\psi should not be bold face

$$P_\delta(\Gamma_{b,k}(z);\xi)=2\pi\sum_{n,r=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}}{n!r!}{}_0\Psi_0\cdot\left[\begin{matrix}-\\-\end{matrix}\middle|k(n-r)\frac{\ln\left[1+(\delta-1)\xi\right]}{\delta-1}\right].\tag{44}$$

Theorem 18. *The E-K fractional transform (of multiplicity m) involving the extended k-gamma function is computed as*

$$\begin{aligned}&\left(\boldsymbol{I}_{(\boldsymbol{\beta}_k),\boldsymbol{m}}^{(\boldsymbol{\gamma}_k),(\boldsymbol{\nu}_k)}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma_{b,k}(z);s\right\}\right)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}{}_m\Psi_m\\&\cdot\left[\begin{matrix}\left(\gamma_i+1+\dfrac{\chi-1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+\delta_i+1+\dfrac{\chi-1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m\end{matrix}\middle|-\frac{e^{k(\ln[1+(\delta-1)\xi]/\delta-1)}}{k}-\frac{b^k e^{-k(\ln[1+(\delta-1)\xi]/\delta-1)}}{k}\right]\\&\cdot\left[-\beta_k(1+\gamma_k)\right]<p;\delta_k\geq0;k=1,\cdots,m.\end{aligned}\tag{45}$$

Proof. Let us first consider

$$\begin{aligned}&\left(\boldsymbol{I}_{(\boldsymbol{\beta}_k),\boldsymbol{m}}^{(\boldsymbol{\gamma}_k),(\boldsymbol{\nu}_k)}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma_{b,k}(z);s\right\}\right)(x)\\&=\left(I_{(\beta k),m}^{(\gamma k),(\delta k)}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}2\pi\right.\\&\cdot\sum_{n,r,p=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}(k)^p(n-r)^p}{n!r!p!}\\&\left.\cdot\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^p\right)(x),\end{aligned}\tag{46}$$

then exchanging the summation and integration

$$\begin{aligned}&\left(\boldsymbol{I}_{(\boldsymbol{\beta}_k),\boldsymbol{m}}^{(\boldsymbol{\gamma}_k),(\boldsymbol{\nu}_k)}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma_{b,k}(z);s\right\}\right)(x)\\&=2\pi\sum_{n,r,p=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}(k)^p(n-r)^p}{n!r!p!}\\&\cdot\left(\boldsymbol{I}_{(\boldsymbol{\beta}_k),\boldsymbol{m}}^{(\boldsymbol{\gamma}_k),(\boldsymbol{\nu}_k)}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^p\right)(x).\end{aligned}\tag{47}$$

Next, to solve the RHS integral $(\boldsymbol{I}_{(\boldsymbol{\beta}_k),\boldsymbol{m}}^{(\boldsymbol{\gamma}_k),(\boldsymbol{\nu}_k)}((\ln[1+(\delta-1)s])/(\delta-1))^{\chi-1}((\ln[1+(\delta-1)s])/(\delta-1))^p)$, we make substitution $u=(\ln[1+(\delta-1)s])/(\delta-1)$; then, using (27) with back substitution $(\ln[1+(\delta-1)s])/(\delta-1)=u$ executes the following:

$$\begin{aligned}&\left(\boldsymbol{I}_{(\boldsymbol{\beta}_k),\boldsymbol{m}}^{(\boldsymbol{\gamma}_k),(\boldsymbol{\nu}_k)}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma_{b,k}(z);s\right\}\right)(x)\\&=2\pi\sum_{n,r,p=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}(k)^p(n-r)^p}{n!r!p!}\\&\cdot\prod_{i=1}^{m}\frac{\Gamma(\gamma_i+1+((\chi+p-1)/\beta_i))}{\Gamma(\gamma i+\delta i+1+((\chi+p-1)/\beta_i))}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{p+\chi-1},\end{aligned}\tag{48}$$

which after modifications by using (18) leads to the following:

$$\begin{aligned}&\left(\boldsymbol{I}_{(\boldsymbol{\beta}_k),\boldsymbol{m}}^{(\boldsymbol{\gamma}_k),(\boldsymbol{\nu}_k)}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma_{b,k}(z);s\right\}\right)\\&=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}\sum_{n,r=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}}{n!r!}{}_m\Psi_m\\&\cdot\left[\begin{matrix}\left(\gamma_i+1+\dfrac{\chi-1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+\delta_i+1+\dfrac{\chi-1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m\end{matrix}\middle|k(n-r)\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)\right]\\&\cdot\left[-\beta_k(1+\gamma_k)\right]<p;\delta_k\geq0;k=1,\cdots,m.\end{aligned}\tag{49}$$

Henceafter, by making use of Lemma 15 leads to the required result. □

Corollary 19. *The E-K fractional transform (of multiplicity m) involving the extended gamma function is computed as*

$$\left(\boldsymbol{I}_{(\beta_k),\boldsymbol{m}}^{(\gamma_k),(\nu_k)}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\{\Gamma_b(\mathrm{z});s\}\right)(\mathrm{x})=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}{}_m\Psi_m\cdot\left[\begin{array}{c}\left(\gamma_i+1+\dfrac{\chi-1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+\delta_i+1+\dfrac{\chi-1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m\end{array}\middle|-\mathrm{e}^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}-b\mathrm{e}^{-(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right].\tag{50}$$

Proof. It follows by considering $k=1$ in (45). □

Corollary 20. *The E-K fractional transform (of multiplicity m) involving the k-gamma function is computed as*

$$\left(\boldsymbol{I}_{(\beta_k),\boldsymbol{m}}^{(\gamma_k),(\nu_k)}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\{\Gamma_k(z);s\}\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}{}_m\Psi_m\cdot\left[\begin{array}{c}\left(\gamma_i+1+\dfrac{\chi-1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+\delta_i+1+\dfrac{\chi-1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m\end{array}\middle|-\frac{e^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right].\tag{51}$$

Proof. It follows by considering $b=0$ in (45). □

Corollary 21. *The E-K fractional transform (of multiplicity m) involving the classical gamma function is computed as*

$$\left(\boldsymbol{I}_{(\beta_k),\boldsymbol{m}}^{(\gamma_k),(\nu_k)}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\{\Gamma(z);s\}\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}{}_m\Psi_m\cdot\left[\begin{array}{c}\left(\gamma_i+1+\dfrac{\chi-1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m\\\left(\gamma_i+\delta_i+1+\dfrac{\chi-1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m\end{array}\middle|-e^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right].\tag{52}$$

Proof. It can be proved by considering $k=1$; $b=0$ in (45). □

Corollary 22. *The E-K fractional transform of multiplicity $m=3$ or the Marichev–Saigo–Maeda fractional integral operator of the extended k-gamma function is given by*

$$\mathrm{I}_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\{\Gamma_{b,k}(\mathrm{z});s\}=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\delta+\chi-\gamma_1-\gamma_1'-1}{}_3\Psi_3\cdot\left[\begin{array}{ccc}(\chi,1)&\left(\chi+\nu-\gamma_1-\gamma_1'-\gamma_2,1\right)&\left(\chi+\gamma_2'-\gamma_1',1\right)\\\left(\chi+\gamma_2',1\right)&\left(\chi+\nu-\gamma_1-\gamma_1',1\right)&\chi+\nu-\gamma_1'-\gamma_2\end{array}\middle|-\frac{\mathrm{e}^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}-\frac{b^k\mathrm{e}^{-k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right].\tag{53}$$

Proof. It can be proved by using the case $m=3$ of Table 1 along with (29) in the main result (45). □

Corollary 23. *The E-K fractional transform of multiplicity $m=3$ or the Marichev–Saigo–Maeda fractional integral* *operator of the extended gamma function is given by*

$$\left(I_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\left(\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\{\Gamma_b(z);s\}\right)\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\delta+\chi-\gamma_1-\gamma_1'-1}{}_3\Psi_3 \cdot\left[\begin{matrix}(\chi,1) & \left(\chi+\nu-\gamma_1-\gamma_1'-\gamma_2,1\right) & \left(\chi+\gamma_2'-\gamma_1',1\right)\\ \left(\chi+\gamma_2',1\right) & \left(\chi+\nu-\gamma_1-\gamma_1',1\right) & \chi+\nu-\gamma_1'-\gamma_2\end{matrix}\middle| -e^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}-be^{-(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right]. \tag{54}$$

Proof. It follows by considering $k=1$ in (53). □

Corollary 24. *The E-K fractional transform of multiplicity $m=3$ or the Marichev–Saigo–Maeda fractional integral operator of the k-gamma function is given by*

$$\left(I_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\{\Gamma_k(z);s\}\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\delta+\chi-\gamma_1-\gamma_1'-1}{}_3\Psi_3 \cdot\left[\begin{matrix}(\chi,1) & \left(\chi+\nu-\gamma_1-\gamma_1'-\gamma_2,1\right) & \left(\chi+\gamma_2'-\gamma_1',1\right)\\ \left(\chi+\gamma_2',1\right) & \left(\chi+\nu-\gamma_1-\gamma_1',1\right) & \chi+\nu-\gamma_1'-\gamma_2\end{matrix}\middle| -\frac{e^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right]. \tag{55}$$

Proof. It follows by considering $b=0$ in (53). □

Corollary 25. *The E-K fractional transform of multiplicity $m=3$ or the Marichev–Saigo–Maeda fractional integral operator of gamma function is given by*

$$\left(I_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\{\Gamma(z);s\}\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\delta+\chi-\gamma_1-\gamma_1'-1}{}_3\Psi_3 \cdot\left[\begin{matrix}(\chi,1) & \left(\chi+\nu-\gamma_1-\gamma_1'-\gamma_2,1\right) & \left(\chi+\gamma_2'-\gamma_1',1\right)\\ \left(\chi+\gamma_2',1\right) & \left(\chi+\nu-\gamma_1-\gamma_1',1\right) & \chi+\nu-\gamma_1'-\gamma_2\end{matrix}\middle| -\frac{e^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right]. \tag{56}$$

Proof. It follows by considering $k=b=0$ in (53). □

Corollary 26. *The subsequent new result is valid containing the Laplace transform of extended k-gamma function:*

$$I_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma_{b,k}(z);s\right\}=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\delta+\chi-\gamma_1-\gamma_1'-1}{}_3\Psi_3 \cdot\left[\begin{matrix}\left(1-\chi-\nu+\gamma_1+\gamma_1',1\right) & \left(1-\chi+\gamma_1+\gamma_2'-\nu,1\right) & 1-\chi-\gamma_1\\ (1-\chi,1) & \left(1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\nu,1\right) & 1-\chi+\gamma_1-\gamma_2\end{matrix}\middle| -\frac{e^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right]. \tag{57}$$

Proof. It can be proved by using case $m = 3$ of Table 1 along with (31) in the main result (45). □

Corollary 27. *The subsequent new result is valid containing the Laplace transform of extended gamma function:*

$$\left(\mathrm{I}_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1} P_\delta\{\Gamma_b(z);s\}\right)(x) = 2\pi\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3 \cdot \left[\begin{matrix}\left(1-\chi-\nu+\gamma_1+\gamma_1',1\right) & \left(1-\chi+\gamma_1+\gamma_2'-\nu,1\right) & 1-\chi-\gamma_1 \\ (1-\chi,1) & \left(1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\nu,1\right) & 1-\chi+\gamma_1-\gamma_2\end{matrix}\middle| -e^{(\ln[1+(\delta-1)\xi])/(\delta-1)} - be^{-(\ln[1+(\delta-1)\xi])/(\delta-1)}\right]. \tag{58}$$

Proof. It follows by considering $k = 1$ in (57). □

Corollary 28. *The subsequent new result is valid containing the Laplace transform of k-gamma function:*

$$\left(\mathrm{I}_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1} P_\delta\{\Gamma_k(z);s\}\right)(x) = 2\pi\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\delta+\chi-\gamma_1-\gamma_1'-1}, {}_3\Psi_3\left[\begin{matrix}\left(1-\chi-\nu+\gamma_1+\gamma_1',1\right) & \left(1-\chi+\gamma_1+\gamma_2'-\nu,1\right) & 1-\chi-\gamma_1 \\ (1-\chi,1) & \left(1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\nu,1\right) & 1-\chi+\gamma_1-\gamma_2\end{matrix}\middle| -\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right]. \tag{59}$$

Proof. It follows by considering $b = 1$ in (57). □

Corollary 29. *The subsequent new result is valid containing the Laplace transform of gamma function:*

$$\left(\mathrm{I}_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1} P_\delta\{\Gamma(z);s\}\right)(x) = 2\pi\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3 \cdot \left[\begin{matrix}\left(1-\chi-\nu+\gamma_1+\gamma_1',1\right) & \left(1-\chi+\gamma_1+\gamma_2'-\nu,1\right) & 1-\chi-\gamma_1 \\ (1-\chi,1) & \left(1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\nu,1\right) & 1-\chi+\gamma_1-\gamma_2\end{matrix}\middle| -e^{(\ln[1+(\delta-1)\xi])/(\delta-1)}\right]. \tag{60}$$

Proof. It follows by considering $k = 1$ in (57). □

For the benefit of the users, further deductions from the main result (45) and in the light of the discussion presented in Section 2.1 are given as follows:

If $(\gamma_1, \gamma_2, \nu \in \mathbb{C} \wedge \Re(\nu) > 0 \wedge \Re(\chi) > \max[0, \Re(\gamma_1-\gamma_2)])$, then the E-K fractional transform of multiplicity $m = 2$ or the *left handed Saigo integral operator* containing the extended k-gamma function and its specific cases are given by

$$\left(I_{0+}^{\gamma_1,\gamma_2,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma_{b,k}(z);s\right\}\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\(\chi-\gamma_2,1) & (\chi+\nu+\gamma_2)\end{matrix}\middle|-\frac{e^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right],$$
$$\left(I_{0+}^{\gamma_1,\gamma_2,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma_{b}(z);s\right\}\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\(\chi-\gamma_2,1) & (\chi+\nu+\gamma_2)\end{matrix}\middle|-e^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}-be^{-(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right],$$
$$\left(I_{0+}^{\gamma_1,\gamma_2,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma_{k}(z);s\right\}\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\(\chi-\gamma_2,1) & (\chi+\nu+\gamma_2)\end{matrix}\middle|-\frac{e^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right],$$
$$\left(I_{0+}^{\gamma_1,\gamma_2,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma(z);s\right\}\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\(\chi-\gamma_2,1) & (\chi+\nu+\gamma_2)\end{matrix}\middle|-e^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right]. \tag{61}$$

If $\gamma_1,\gamma_2,\nu\in\mathbb{C}\wedge\Re(\nu)>0\wedge\Re(\chi)<1+\min\left[\Re(\gamma_1),\Re(\gamma_2)\right]$, then the E-K fractional transform of multiplicity $m=2$ or *the right handed Saigo integral operator* containing the extended k-gamma function, and its special cases are given by

$$\left(I_{-}^{\gamma_1,\gamma_2,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma_{b,k}(z);s\right\}\right)(x)$$
$$=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\gamma_1-\chi+1,1) & (\gamma_2-\chi+1,1)\\(1-\chi,1) & ((\gamma_1+\gamma_2+\nu-\chi+1,1)\end{matrix}\middle|-\frac{e^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right],$$

$$\left(I_{-}^{\gamma_1,\gamma_2,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma_{b}(z);s\right\}\right)(x)$$
$$=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\gamma_1-\chi+1,1) & (\gamma_2-\chi+1,1)\\(1-\chi,1) & ((\gamma_1+\gamma_2+\nu-\chi+1,1)\end{matrix}\middle|-e^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}-be^{-(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right],$$

$$\left(I_{-}^{\gamma_1,\gamma_2,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma_{k}(z);s\right\}\right)(x)$$
$$=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\gamma_1-\chi+1,1) & (\gamma_2-\chi+1,1)\\(1-\chi,1) & ((\gamma_1+\gamma_2+\nu-\chi+1,1)\end{matrix}\middle|-\frac{e^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right],$$

$$\left(I_{-}^{\gamma_1,\gamma_2,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_\delta\left\{\Gamma(z);s\right\}\right)(x)$$
$$=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\gamma_1-\chi+1,1) & (\gamma_2-\chi+1,1)\\(1-\chi,1) & ((\gamma_1+\gamma_2+\nu-\chi+1,1)\end{matrix}\middle|-e^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right]. \tag{62}$$

If $\gamma, \nu \in \mathbb{C}, \Re(\nu) > 0, \Re(\nu + \mathrm{p}) > -\Re(\gamma)$, then the E-K fractional transform of multiplicity $m = 1$ or the *left handed Erdélyi–Kober integral operator* containing the extended k-gamma function, and its special cases are given by

$$\begin{aligned}
&\left(\mathrm{I}_{0+}^{\gamma,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} P_\delta\left\{\Gamma_{b,k}(\mathrm{z});s\right\}\right)(x) = 2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\(\chi+\gamma+\nu,1)\end{matrix}\middle| -\frac{\mathrm{e}^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k} - \frac{b^k \mathrm{e}^{-k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right],\\
&\left(\mathrm{I}_{0+}^{\gamma,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} P_\delta\left\{\Gamma_{b}(\mathrm{z});s\right\}\right)(x) = 2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} {}_2\Psi_2\left[\begin{matrix}(\chi+\gamma,1)\\(\chi+\gamma+\nu,1)\end{matrix}\middle| -\mathrm{e}^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)} - b\mathrm{e}^{-(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right],\\
&\left(\mathrm{I}_{0+}^{\gamma,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} P_\delta\left\{\Gamma_{k}(\mathrm{z});s\right\}\right)(x) = 2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\(\chi+\gamma+\nu,1)\end{matrix}\middle| -\frac{\mathrm{e}^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right],\\
&\left(\mathrm{I}_{0+}^{\gamma,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} P_\delta\left\{\Gamma(\mathrm{z});s\right\}\right)(x) = 2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\(\chi+\gamma+\nu,1)\end{matrix}\middle| -\mathrm{e}^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right].
\end{aligned} \tag{63}$$

If $\gamma, \nu \in \mathbb{C}; \Re(\chi + \mathrm{p}) < 1 + \Re(\gamma)$, then the E-K fractional transform of multiplicity $m = 1$ or the *right handed Erdélyi–Kober integral operator* containing the extended k-gamma function, and its special cases are given by

$$\begin{aligned}
&\left(\mathrm{I}_{0-}^{\gamma,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} P_\delta\left\{\Gamma_{b,k}(\mathrm{z});s\right\}\right)(\mathrm{x}) = 2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\(\gamma+\nu-\chi+1,-1)\end{matrix}\middle| -\frac{\mathrm{e}^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k} - \frac{b^k \mathrm{e}^{-k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right],\\
&\left(\mathrm{I}_{0-}^{\gamma,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} P_\delta\left\{\Gamma_{b}(\mathrm{z});s\right\}\right)(x) = 2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\(\gamma+\nu-\chi+1,-1)\end{matrix}\middle| -\mathrm{e}^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)} - b\mathrm{e}^{-(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right],\\
&\left(\mathrm{I}_{0-}^{\gamma,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} P_\delta\left\{\Gamma_{k}(\mathrm{z});s\right\}\right)(x) = 2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} \left[\begin{matrix}(\gamma-\chi+1,-1)\\(\gamma+\nu-\chi+1,-1)\end{matrix}\middle| -\frac{\mathrm{e}^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right],\\
&\left(\mathrm{I}_{0-}^{\gamma,\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} P_\delta\left\{\Gamma(\mathrm{z});s\right\}\right)(x) = 2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\(\gamma+\nu-\chi+1,-1)\end{matrix}\middle| -\mathrm{e}^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right].
\end{aligned} \tag{64}$$

If $\nu, \chi \in \mathbb{C}, \Re(\nu) > 0, \Re(\chi) > 0$, then the E-K fractional transform of multiplicity $m = 1$ or the *left handed Riemann-Liouville (R-L)* integral operator containing the extended k-gamma function, and its special cases are given by

$$\begin{aligned}
&\left(\mathrm{I}_{0+}^{\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} P_\delta\left\{\Gamma_{b,k}(\mathrm{z});s\right\}\right)(\mathrm{x}) = 2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi+\delta-1} {}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\nu+\chi,1)\end{matrix}\middle| -\frac{\mathrm{e}^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k} - \frac{b^k \mathrm{e}^{-k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right],\\
&\left(\mathrm{I}_{0+}^{\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} P_\delta\left\{\Gamma_{b}(\mathrm{z});s\right\}\right)(\mathrm{x}) = 2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi+\delta-1} {}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\nu+\chi,1)\end{matrix}\middle| -\mathrm{e}^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)} - b\mathrm{e}^{-(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right],\\
&\left(\mathrm{I}_{0+}^{\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} P_\delta\left\{\Gamma_{k}(\mathrm{z});s\right\}\right)(x) = 2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi+\delta-1} {}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\nu+\chi,1)\end{matrix}\middle| -\frac{\mathrm{e}^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right],\\
&\left(\mathrm{I}_{0+}^{\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1} P_\delta\left\{\Gamma(\mathrm{z});s\right\}\right)(x) = 2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi+\delta-1} {}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\nu+\chi,1)\end{matrix}\middle| -\mathrm{e}^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right].
\end{aligned} \tag{65}$$

If $\delta, \chi \in \mathbb{C}, 0 < \Re(\delta) < 1 - \Re(\chi + p)$, then the E-K fractional transform of multiplicity $m = 1$ or *the right handed Riemann-Liouville (R-L) integral operator* containing the extended k-gamma function, and its special cases are given by

$$\left(\mathrm{I}_{-}^{\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_{\delta}\{\Gamma_{b,k}(z);s\}\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi+\delta-1}{}_{1}\Psi_{1}\left[\begin{matrix}(1-\nu-\chi,-1)\\(1-\chi,-1)\end{matrix}\middle|-\frac{e^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}-\frac{b^{k}e^{-k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right],$$
$$\left(\mathrm{I}_{-}^{\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_{\delta}\{\Gamma_{b}(z);s\}\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi+\delta-1}{}_{1}\Psi_{1}\left[\begin{matrix}(1-\nu-\chi,-1)\\(1-\chi,-1)\end{matrix}\middle|-e^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}-be^{-(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right],$$
$$\left(\mathrm{I}_{-}^{\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_{\delta}\{\Gamma_{k}(z);s\}\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi+\delta-1}{}_{1}\Psi_{1}\left[\begin{matrix}(1-\nu-\chi,-1)\\(1-\chi,-1)\end{matrix}\middle|-\frac{e^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right],$$
$$\left(\mathrm{I}_{-}^{\nu}\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}P_{\delta}\{\Gamma(z);s\}\right)(x)=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi+\delta-1}{}_{1}\Psi_{1}\left[\begin{matrix}(1-\nu-\chi,-1)\\(1-\chi,-1)\end{matrix}\middle|-e^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right].\tag{66}$$

It is to be remarked that the results obtained for this case i-e $m = 1$ can be expressed in terms of Mittag-Leffler function by using the relation

$$\begin{aligned}&{}_{1}\Psi_{1}\left[\begin{matrix}(\chi,1)\\(\nu+\chi,1)\end{matrix}\middle|k(\mathrm{n}-\mathrm{r})\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)\right]\\&=\Gamma(\chi)\mathrm{E}_{1,\nu+\chi}^{\chi}\left(k(\mathrm{n}-\mathrm{r})\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)\right).\end{aligned}\tag{67}$$

*3.2. Generalized Fractional Derivatives Containing the Extended **k**-Gamma Function.* By using the methodology of Theorem 18 and new representation of the extended $\boldsymbol{k}$ -gamma function, we can find the multiple fractional derivatives concerning the extended $\boldsymbol{k}$-gamma function. Here, we obtain them directly by using the general result [46, Theorem 4] also given as

$$\begin{aligned}&D_{(\beta k),m}^{(\gamma k)_1^m,(\nu k)}\left\{z^{c}{}_{p}\Psi_{q}\left[\begin{matrix}(a_i,\alpha_i)_1^p\\(b_j,\beta_j)_1^q\end{matrix};\lambda z^{\mu}\right]\right\}\\&=z^{c}\left\{{}_{p+m}\Psi_{q+m}\left[\begin{matrix}(a_i,\alpha_i)_1^p,\left(\gamma_k+\nu_k+1+\dfrac{c}{\beta_k},\dfrac{1}{\beta_k}\right)_1^m\\(b_j,\beta_j)_1^q,\left(\gamma_k+1+\dfrac{c}{\beta_k},\dfrac{1}{\beta_k}\right)_1^m\end{matrix};\lambda z^{\mu}\right]\right\},\end{aligned}\tag{68}$$

Generalized fractional derivatives containing the extended k-gamma function is obtained by applying (68) on (42):

$$\begin{aligned}D_{(\beta k),m}^{(\gamma k)_1^m,(\nu k)}\left\{\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1}P_{\delta}\{\Gamma_{b,k}(z);s\}\right\}&=2\pi\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1}{}_{m}\Psi_{m}\\&\cdot\left[\begin{matrix}\left(\gamma_k+\nu_k+1+\dfrac{c}{\beta_k},\dfrac{1}{\beta_k}\right)_1^m\\\left(\gamma_k+1+\dfrac{c}{\beta_k},\dfrac{1}{\beta_k}\right)_1^m\end{matrix}\middle|-\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}-\frac{b^{k}e^{-k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right].\end{aligned}\tag{69}$$

Similarly, one can obtain the corresponding Marichev–Saigo–Maeda fractional derivatives [53–55] as follows:

$$D_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\left(\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1}P_\delta\{\Gamma_{b,k}(z);s\}\right)$$
$$=2\pi\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1}{}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi-\gamma_2+\gamma_1,1) & \left(\chi+\gamma_{1+}\gamma_1'+\gamma_2'-\nu,1\right)\\ (\chi-\gamma_2,1) & \left(\chi-\nu+\gamma_1++\gamma_2',1\right) & \left(\chi-\nu+\gamma_1'+\gamma_1,1\right)\end{matrix}\middle|-\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right],$$
$$D_{-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\nu}\left(\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1}P_\delta\{\Gamma_{b,k}(z);s\}\right)$$
$$=2\pi\omega^{\chi-1}{}_3\Psi_3\left[\begin{matrix}\left(1-\chi+\gamma_2',1\right) & \left(1+\gamma_2'-\chi-\gamma_2+\gamma_1,1\right) & \left(1-\chi-\gamma_{1-}\gamma_1'+\nu,1\right)\\ (1-\chi,1) & \left(1-\chi-\gamma_1'+\gamma_2',1\right) & \left(1-\chi+\nu-\gamma_1'-\gamma_1-\gamma_2,1\right)\end{matrix}\middle|-\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right]. \tag{70}$$

The corresponding Saigo fractional derivatives [53–55] are given as follows:

$$D_{0+}^{\gamma_1,\gamma_2,\nu}\left(\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1}P_\delta\{\Gamma_{b,k}(z);s\}\right)$$
$$=2\pi\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\nu+\gamma_2+\gamma_1,1)\\ (\chi+\gamma_2,1) & (\chi+\nu,1)\end{matrix}\middle|-\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right],$$
$$D_{-}^{\gamma_1,\gamma_2,\nu}\left(\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1}P_\delta\{\Gamma_{b,k}(z);s\}\right)$$
$$=2\pi\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(1-\chi-\gamma_2,1) & (1-\chi+\nu+\gamma_1,1)\\ (1-\chi+\nu-\gamma_2,1) & (1-\chi,1)\end{matrix}\middle|-\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right]. \tag{71}$$

For $\boldsymbol{\beta_k}=\boldsymbol{\beta}$ in (69), the following generalized Erdélyi–Kober fractional derivative is found:

$$D_{\beta}^{\gamma,\nu}\left\{\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1}P_\delta\{\Gamma_{b,k}(z);s\}\right\}=2\pi\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1}{}_1\Psi_1\left[\begin{matrix}\left(\gamma+\nu+1+\frac{c}{\beta},\frac{1}{\beta}\right)\\ \left(\gamma+1+\frac{c}{\beta},\frac{1}{\beta}\right)\end{matrix}\middle|-\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right]. \tag{72}$$

Similarly, one can obtain the corresponding left and right-handed E–K and R-L fractional derivatives [53–55] as follows:

$$D_{0+}^{\gamma,\nu}\left\{\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1} P_\delta\{\Gamma_{b,k}(z);s\}\right\} = 2\pi\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\gamma+\nu+\chi,1)\\(\gamma+\chi,1)\end{matrix}\middle| -\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right], \tag{73}$$

$$D_{-}^{\gamma,\nu}\left\{\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1} P_\delta\{\Gamma_{b,k}(z);s\}\right\} = 2\pi\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(1-\chi+\gamma+\nu,1)\\(1-\chi+\gamma,1)\end{matrix}\middle| -\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right] \tag{74}$$

$$D_{0+}^{\nu}\left\{\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1} P_\delta\{\Gamma_{b,k}(z);s\}\right\} = 2\pi\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1-\delta} {}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\chi-\nu,1)\end{matrix}\middle| -\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right], \tag{75}$$

$$D_{-}^{\nu}\left\{\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1} P_\delta\{\Gamma_{b,k}(z);s\}\right\} = 2\pi\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)^{\chi-1-\delta} {}_1\Psi_1\left[\begin{matrix}(\nu-\chi+1,1)\\(1-\chi,1)\end{matrix}\middle| -\frac{e^{k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}-\frac{b^k e^{-k(\ln[1+(\delta-1)\xi])/(\delta-1)}}{k}\right]. \tag{76}$$

Remark 30. Special cases:

(1) By taking $b=0$ in the above results (69)-(76), the corresponding fractional derivatives of generalized gamma function can be attained

(2) By taking $k=1$ in the above results (69)-(76), the corresponding fractional derivatives of k-gamma function can be attained

(3) Similarly, by considering $b=0$ and $k=1$; in the above results (69)-(76), the corresponding fractional derivatives of classical gamma function can be attained

Remark 31. For the interest of large audience, the fractional integrals and derivatives involving the Laplace transform of the extended k-gamma function are listed in the Appendices A and B.

3.3. Pathway Transforms and the Solution of Fractional Kinetic Equation Involving the Extended k-Gamma Function. The use of non-integer operators has appeared recently in various technical [59–61] and science specialties [17–19]. For instance, the fractional kinetic equation is useful in studying gas theory, astrophysics, and aerodynamics [62–64]. A general non-integer order kinetic equation containing the extended k-gamma function was recently formulated and solved [34] using the Laplace transform. The main goal of this section is to formulate and solve this problem using the pathway transform and then validate the results using both methods.

Sexana et al. [62] used subsequent kinetic equations to analyze the reaction and destruction to model changes in production rates:

$$\frac{d\Theta}{dt} = -d(\Theta_t) + p(\Theta_t)\,; \quad \begin{matrix}\Theta_t(t^*) = \Theta(t-t^*),\ t^*>0\\ \Theta=\Theta(t) = \text{change in reaction}\\ d=d(\Theta)=\text{Change in destruction}\\ p=p(\Theta)=\text{Change in production}\end{matrix}. \tag{77}$$

If we neglect the inhomogeneity and spatial fluctuation of $\Theta(t)$ with the concentration of species, $\Theta(t=0)=\Theta_0$, then we can rewrite it as

$$\frac{d\Theta_j}{dt} = -c_j\Theta_j(t), \tag{78}$$

and further to this, we ignore the subscript j and integrate (78) to get

$$\Theta(t)-\Theta_0 = -c\,I_{0+}^{-1}\Theta(t), \tag{79}$$

with c as a constant. Haubold and Mathai [59] developed the non-integer order kinetic equation:

$$\Theta(t)-\Theta_0 = -c^{\delta} I_{0+}^{\delta}\Theta(t), \tag{80}$$

by using the Riemann–Liouville (R-L) fractional integral ($I_{0+}^{\delta}, \delta>0$). We now formulate and solve the fractional kinetic equation, as proposed by Haubold and Mathai [59] i-e for any integrable function $f(t)$:

$$\Theta(t) - f(t)\Theta_0 = -d^{\delta} I_{0+}^{\nu}\Theta(t). \tag{81}$$

Now, we can formulate and solve the following fractional kinetic equation comprising of the extended k-gamma function by using the above information.

Theorem 32. *By using pathway transform, the solution of non-integer kinetic equation*

$$\Theta(t)-\Theta_0\Gamma_{b,k}(t)=-d^{\nu}\,I_{0+}^{\nu}\Theta(t)\,;b,k,d>0,\nu>0 \tag{82}$$

is computed as follows:

$$\Theta(t)=\frac{2\pi\chi_0}{t}\sum_{n,r,p=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}((k(n-r))/t)^p E_{\nu,-p}(-d^{\nu}t^{\nu})}{n!r!p!}. \tag{83}$$

Proof. By applying the path way transform P_δ on (82)

$$P_\delta\{\Theta(t)\}-\Theta_0 P_\delta\{\Gamma_{\mathrm{b,k}}(\mathrm{t})\}=P_\delta\{-d^{\nu}\,I_{0+}^{\nu}\Theta(t)\}, \tag{84}$$

as well as by using (4) and (42) yield the following:

$$\Theta(s)=2\pi\chi_0\sum_{\mathrm{n,r,p}=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}(\mathrm{k})^{\mathrm{p}}(\mathrm{n-r})^{\mathrm{p}}}{\mathrm{n!r!p!}}\cdot\left(\frac{\delta-1}{\ln\left[1+(\delta-1)s\right]}\right)^{p}-\left(\frac{(\delta-1)/(\ln\left[1+(\delta-1)s\right])}{d}\right)^{-\nu}\Theta(s),$$

$$\Theta(s)\left[1+\left(\frac{(\delta-1)/(\ln\left[1+(\delta-1)s\right])}{d}\right)^{-\nu}\right]=2\pi\Theta_0\sum_{\mathrm{n,r,p}=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}(\mathrm{k})^{\mathrm{p}}(\mathrm{n-r})^{\mathrm{p}}}{\mathrm{n!r!p!}}\cdot\{(\ln\left[1+(\delta-1)s\right])/(\delta-1)\}^{p}. \tag{85}$$

After some simple calculation, one can obtain

$$\Theta(s)=2\pi\Theta_0\sum_{\mathrm{n,r,p}=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}(\mathrm{k})^{\mathrm{p}}(\mathrm{n-r})^{\mathrm{p}}}{\mathrm{n!r!p!}}\times\{(\ln\left[1+(\delta-1)s\right])/(\delta-1)\}^{p}\sum_{m=0}^{\infty}\cdot\left[-\left(\frac{(\ln\left[1+(\delta-1)s\right])/(\delta-1)}{d}\right)^{-\nu}\right]^{m}\Gamma(\nu). \tag{86}$$

The inverse path way transform of (86) is computed by using (3) and given as follows:

$$\Theta(t)=2\pi\Theta_0\sum_{\mathrm{n,r,p}=0}^{\infty}\frac{(-1/k)^{n+r}(b)^{kr}(\mathrm{k})^{\mathrm{p}}(\mathrm{n-r})^{\mathrm{p}}}{\mathrm{n!r!p!}}t^{-p-1}\times\sum_{\mathrm{m}=0}^{\infty}\frac{(-\mathrm{d}^{\nu}\mathrm{t}^{\nu})^{\mathrm{m}}}{\Gamma(\nu\mathrm{m}-\mathrm{p})}\,;\nu\mathrm{m}-\mathrm{p}>0\,;\nu>0. \tag{87}$$

□

As a result, using (17) in (87) yields the desired result. Special cases:

(1) By taking $b=0$ in the above results (82)-(83), the corresponding non-integer kinetic equation containing $\Gamma_{\mathrm{k}}(\mathrm{t})$ (k-gamma function) as well as the solution can be obtained [34]

(2) By taking $k=1$ in the above results (82)-(83), the corresponding non-integer kinetic equation containing $\Gamma_{\mathrm{b}}(\mathrm{t})$ (generalized gamma function) and its solution can be obtained [34]

(3) Similarly, by considering $b=0$ and $k=1$, in the above results (82)-(83), the corresponding non-integer kinetic equation containing $\Gamma(\mathrm{t})$ (gamma function) and its solution can be obtained [34]

Remark 33. Hence, we obtain the same solution [34] by using pathway transform. It validates that approach is consistent by using both transforms. Similarly, the results under various other related transforms like Sumudu transform [65], Natural transform [66], and Elzaki transform [67] can be obtained and validated.

3.4. New Integrals of Products Involving Special Functions. It is worth noting that the subsequent results containing the products of a large class of special functions are evaluated by taking (42) and (45):

$$\int_0^1\left(\frac{\ln\left[1+(\delta-1)\xi\right]}{\delta-1}\right)^{\chi-1}\exp\left(-\frac{\mathrm{e}^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}-\frac{b^k\mathrm{e}^{-k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right)\cdot H_{m,m}^{m,0}\left[\xi\left|\begin{matrix}\left(\gamma_i+\nu_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right.\right]d\xi=2\pi\left(\frac{\ln\left[1+(\delta-1)s\right]}{\delta-1}\right)^{\chi-1}{}_m\Psi_m\cdot\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\gamma_i+\nu_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\left|-\frac{\mathrm{e}^{k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}-\frac{b^k\mathrm{e}^{-k(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}}{k}\right.\right], \tag{88}$$

and for $k=1\,;b=0$ special cases can be obtained:

$$\int_0^1\left(\frac{\ln\left[1+(\delta-1)\xi\right]}{\delta-1}\right)^{\chi-1}\exp\left(-\mathrm{e}^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right)\cdot H_{m,m}^{m,0}\left[\xi\left|\begin{matrix}\left(\gamma_i+\nu_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\right.\right]d\xi=2\pi\left(\frac{\ln\left[1+(\delta-1)\xi\right]}{\delta-1}\right)^{\chi-1}\cdot{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\\ \left(\gamma_i+\nu_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\left|-\mathrm{e}^{(\ln\left[1+(\delta-1)\xi\right])/(\delta-1)}\right.\right]. \tag{89}$$

By making use of (13) and (23) alongwith the definition of Dirac delta function, subsequent new integrals of products of special functions can be computed:

$$\int_{\xi\epsilon\mathbb{C}} \Gamma_{b,k}(z\xi) H^{m,0}_{m,m}\left[\xi \middle| \begin{matrix} \left(\gamma_i+\nu_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \end{matrix}\right] d\xi = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1/k)^{n+r}(b)^{kr}}{n!r!} \int_{\xi\epsilon\mathbb{C}} \delta(z\xi+\mathrm{nk}-\mathrm{kr}) \cdot H^{m,0}_{m,m}\left[\xi \middle| \begin{matrix} \left(\gamma_i+\nu_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \end{matrix}\right] d\xi = 2\pi \mathrm{z}^{-1} \sum_{n,r=0}^{\infty} \frac{(-1/k)^{n+r}(b)^{kr}}{n!r!} \cdot H^{m,0}_{m,m}\left[\frac{\mathrm{k(r-n)}}{z} \middle| \begin{matrix} \left(\gamma_i+\nu_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \end{matrix}\right] = \frac{2\pi}{z} H^{m,0}_{m,m}\left[-\frac{\mathrm{e}^{k/z}}{k}-\frac{b^k\mathrm{e}^{-k/z}}{k} \middle| \begin{matrix} \left(\gamma_i+\nu_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \end{matrix}\right]. \tag{90}$$

Hence, the following special cases involving the family of gamma function can be obtained:

$$\int_{\xi\epsilon\mathbb{C}} \Gamma_{b}(z\xi) H^{m,0}_{m,m}\left[\xi \middle| \begin{matrix} \left(\gamma_i+\nu_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \end{matrix}\right] d\xi = \frac{2\pi}{z} H^{m,0}_{m,m}\left[-\mathrm{e}^{1/z}-b\mathrm{e}^{-1/z} \middle| \begin{matrix} \left(\gamma_i+\nu_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \end{matrix}\right],$$

$$\int_{\xi\epsilon\mathbb{C}} \Gamma_{k}(z\xi) H^{m,0}_{m,m}\left[\xi \middle| \begin{matrix} \left(\gamma_i+\nu_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \end{matrix}\right] d\xi = \frac{2\pi}{z} H^{m,0}_{m,m}\left[-\frac{\mathrm{e}^{k/z}}{k} \middle| \begin{matrix} \left(\gamma_i+\nu_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \end{matrix}\right],$$

$$\int_{\xi\epsilon\mathbb{C}} \Gamma(z\xi) H^{m,0}_{m,m}\left[\xi \middle| \begin{matrix} \left(\gamma_i+\nu_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \end{matrix}\right] d\xi = \frac{2\pi}{z} H^{m,0}_{m,m}\left[-\mathrm{e}^{1/z} \middle| \begin{matrix} \left(\gamma_i+\nu_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+1-\dfrac{1}{\beta_i},\dfrac{1}{\beta_i}\right)_1^m \end{matrix}\right], \tag{91}$$

and further new integrals of products of special functions are computable by using the relation of Fox-H function $H^{m,0}_{m,m}[(\xi/z) \big| \begin{smallmatrix} (\gamma_k+\nu_k+1-1/\beta k, (1/\beta k))_1^m \\ (\gamma_k+1-1/\beta k, (1/\beta k))_1^m \end{smallmatrix}]$ with other special functions as mentioned for example in Equations (19)-(20) for G-function, Fox-Wright function, and Mittag-Leffler function. For example,

$$\int_{\xi\epsilon\mathbb{C}} \Gamma_{b,k}(z\xi) G^{m,0}_{m,m}\left[\xi \middle| \begin{matrix} (\gamma_i+\nu_i)_1^m \\ (\gamma_i)_1^m \end{matrix}\right] d\xi = \frac{2\pi}{z} G^{m,0}_{m,m}\left[-\frac{\mathrm{e}^{k/z}}{k}-\frac{b^k\mathrm{e}^{-k/z}}{k} \middle| \begin{matrix} (\gamma_i+\nu_i)_1^m \\ (\gamma_i)_1^m \end{matrix}\right],$$

$$\int_{\xi\epsilon\mathbb{C}} \Gamma_{b}(z\xi) G^{m,0}_{m,m}\left[\xi \middle| \begin{matrix} (\gamma_i+\nu_i)_1^m \\ (\gamma_i)_1^m \end{matrix}\right] d\xi = \frac{2\pi}{z} G^{m,0}_{m,m}\left[-\mathrm{e}^{1/z}-b\mathrm{e}^{-1/z} \middle| \begin{matrix} (\gamma_i+\nu_i)_1^m \\ (\gamma_i)_1^m \end{matrix}\right],$$

$$\int_{\xi\epsilon\mathbb{C}} \Gamma_{k}(z\xi) G^{m,0}_{m,m}\left[\xi \middle| \begin{matrix} (\gamma_i+\nu_i)_1^m \\ (\gamma_i)_1^m \end{matrix}\right] d\xi = \frac{2\pi}{z} G^{m,0}_{m,m}\left[-\frac{\mathrm{e}^{k/z}}{k} \middle| \begin{matrix} (\gamma_i+\nu_i)_1^m \\ (\gamma_i)_1^m \end{matrix}\right],$$

$$\int_{\xi\epsilon\mathbb{C}} \Gamma(z\xi) G^{m,0}_{m,m}\left[\xi \middle| \begin{matrix} (\gamma_i+\nu_i)_1^m \\ (\gamma_i)_1^m \end{matrix}\right] d\xi = \frac{2\pi}{z} G^{m,0}_{m,m}\left[-\mathrm{e}^{1/z} \middle| \begin{matrix} (\gamma_i+\nu_i)_1^m \\ (\gamma_i)_1^m \end{matrix}\right]. \tag{92}$$

Similarly, ([64], Equation (32)) can be rewritten as by using (23):

$$\int_0^1 \frac{(z\omega)^{\rho-1}}{\exp(e^{z\omega})-1} H_{m,m}^{m,0}\left[\omega \left| \begin{matrix} \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \end{matrix}\right.\right] d\omega$$
$$= \omega^{-1}\int_0^\omega \frac{z^{\rho-1}}{\exp(e^{z})-1} H_{m,m}^{m,0}\left[\frac{z}{\omega} \left| \begin{matrix} \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \end{matrix}\right.\right] dz$$
$$= \omega^{\rho-1}\sum_{n=0}^{\infty} {}_m\Psi_m\left[\begin{matrix} \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \end{matrix} \left| -(n+1)e^{\omega} \right.\right]. \tag{93}$$

4. Conclusion

The novel fractional transitions of the extended k-gamma function are procured using the generalized fractional calculus' multiple E-K operators. As a result, specific new images are acquired as special cases for the numerous different other popular fractional transforms. It is only possible because recent research [34] investigates the Laplace transform of various extensions of the gamma function. Furthermore, the pathway transform is used to solve the fractional kinetic equation encompassing these extensions. As corollaries, specific cases involving the gamma function family are discussed. The findings are supported by the previously obtained solution of the fractional kinetic equation involving such functions, which was solved using the Laplace transform. A newly discovered representation of the generalizations of the gamma function and their Laplace transform played an important role. Novel identities containing the Fox-Wright function were found to be extremely useful in simplifying the results. Consequently, numerous results, including [46, Lemmas 7–15, Theorems 2-4], are applicable to the Laplace transform of the gamma function, and the key result (45) along with important cases are completely provable using the known theory and techniques. Given the known representations of the gamma function (as well as its extensions), it is clear that this theory and techniques cannot be applied. As a result, it is concluded that the results of this article are only possible because of a new representation [34] of as a series of complex delta functions, and this research is significant in expanding the applicability of the gamma function (as well as its extensions) beyond its original domain. By means of this novel definition of the k-gamma functions, we can discover more integrals in a simple way. For example, considering the domain $0<\Re(z)<1$, and for $k=2$, we have corrected an integral of Gaussian function [35, Section 4]:

$$\begin{aligned}\langle \Gamma_2(\mathrm{z}),\zeta(\mathrm{z})\rangle &= 2\pi\sum_{\mathrm{n}=0}^{\infty}\frac{(-1/2)^{\mathrm{n}}}{\mathrm{n}!}\langle \delta(\mathrm{z}+2\mathrm{n}),\zeta(\mathrm{z})\rangle \\ &= 2\pi\sum_{\mathrm{n}=0}^{\infty}\frac{(-1/2)^{\mathrm{n}}}{\mathrm{n}!}\zeta(-2\mathrm{n})=-\pi.\end{aligned} \tag{94}$$

It can be concluded that this research is substantial to enhance the application of extended k-gamma beyond its original domain.

Appendix

A. Generalized Fractional Integrals Involving the Laplace Transform of the Extended k-Gamma Function

The E-K fractional transform of multiplicity m containing the Laplace transform of the extended k-gamma *function* is given by

$$\left(I_{(\beta k),m}^{(\gamma k),(\delta k)} s^{\chi-1}L\{\Gamma_{b,k}(\mathrm{z});s\}\right) = 2\pi s^{\chi-1}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\left| -\frac{\mathrm{e}^{k\xi}}{k}-\frac{b^k\mathrm{e}^{-k\xi}}{k}\right.\right], \tag{A.1}$$

$$\left(I_{(\beta k),m}^{(\gamma k),(\delta k)} s^{\chi-1}L\{\Gamma_{b,k}(\mathrm{z});s\}\right) = 2\pi s^{\chi-1}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\left| -\mathrm{e}^{\xi}-b\mathrm{e}^{-\xi}\right.\right], \tag{A.2}$$

$$\left(I_{(\beta k),m}^{(\gamma k),(\delta k)} s^{\chi-1}L\{\Gamma_{k}(\mathrm{z});s\}\right) = 2\pi s^{\chi-1}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\left| -\frac{\mathrm{e}^{k\xi}}{k}\right.\right], \tag{A.3}$$

$$\left(I_{(\beta k),m}^{(\gamma k),(\delta k)} s^{\chi-1}L\{\Gamma(\mathrm{z});s\}\right) = 2\pi s^{\chi-1x}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m \\ \left(\gamma_i+\delta_i+1+\frac{\chi-1}{\beta_i},\frac{1}{\beta_i}\right)_1^m\end{matrix}\left| -\mathrm{e}^{\xi}\right.\right]. \tag{A.4}$$

The E-K fractional transform of multiplicity $m=3$ or the Marichev–Saigo–Maeda fractional integral operator containing the Laplace transform of the extended k-gamma *function* is given by

$$I_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma_{b,k}(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,1) & (\chi+\gamma_2'-\gamma_1',1)\\ (\chi+\gamma_2',1) & (\chi+\delta-\gamma_1-\gamma_1',1) & \chi+\delta-\gamma_1'-\gamma_2\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{A.5}$$

$$I_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma_{b}(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,1) & (\chi+\gamma_2'-\gamma_1',1)\\ (\chi+\gamma_2',1) & (\chi+\delta-\gamma_1-\gamma_1',1) & \chi+\delta-\gamma_1'-\gamma_2\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{A.6}$$

$$I_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma_{k}(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,1) & (\chi+\gamma_2'-\gamma_1',1)\\ (\chi+\gamma_2',1) & (\chi+\delta-\gamma_1-\gamma_1',1) & \chi+\delta-\gamma_1'-\gamma_2\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{A.7}$$

$$I_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi+\delta-\gamma_1-\gamma_1'-\gamma_2,1) & (\chi+\gamma_2'-\gamma_1',1)\\ (\chi+\gamma_2',1) & (\chi+\delta-\gamma_1-\gamma_1',1) & \chi+\delta-\gamma_1'-\gamma_2\end{matrix}\middle| -e^{\xi}\right],$$

$$I_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma_{b,k}(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} \tag{A.8}$$

$${}_3\Psi_3\left[\begin{matrix}(1-\chi-\delta+\gamma_1+\gamma_1',1) & (1-\chi+\gamma_1+\gamma_2'-\delta,1) & 1-\chi-\gamma_1\\ (1-\chi,1) & (1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\delta,1) & 1-\chi+\gamma_1-\gamma_2\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{A.9}$$

$$I_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma_{b}(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}(1-\chi-\delta+\gamma_1+\gamma_1',1) & (1-\chi+\gamma_1+\gamma_2'-\delta,1) & 1-\chi-\gamma_1\\ (1-\chi,1) & (1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\delta,1) & 1-\chi+\gamma_1-\gamma_2\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{A.10}$$

$$I_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma_{k}(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}(1-\chi-\delta+\gamma_1+\gamma_1',1) & (1-\chi+\gamma_1+\gamma_2'-\delta,1) & 1-\chi-\gamma_1\\ (1-\chi,1) & (1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\delta,1) & 1-\chi+\gamma_1-\gamma_2\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{A.11}$$

$$I_{0-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}(1-\chi-\delta+\gamma_1+\gamma_1',1) & (1-\chi+\gamma_1+\gamma_2'-\delta,1) & 1-\chi-\gamma_1\\ (1-\chi,1) & (1-\chi+\gamma_1+\gamma_1'+\gamma_2+\gamma_2'-\delta,1) & 1-\chi+\gamma_1-\gamma_2\end{matrix}\middle| -e^{\xi}\right]. \tag{A.12}$$

The E-K fractional transform of multiplicity $m = 2$ or the Saigo fractional integral operator containing the Laplace transform of the extended k-gamma *function* is given by

$$I_{0+}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma_{b,k}(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\ (\chi-\gamma_2,1) & (\chi+\delta+\gamma_2)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{A.13}$$

$$I_{0+}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma_b(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\ (\chi-\gamma_2,1) & (\chi+\delta+\gamma_2)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{A.14}$$

$$I_{0+}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma_k(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\ (\chi-\gamma_2,1) & (\chi+\delta+\gamma_2)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{A.15}$$

$$I_{0+}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\gamma_2-\gamma_1,1)\\ (\chi-\gamma_2,1) & (\chi+\delta+\gamma_2)\end{matrix}\middle| -e^{\xi}\right], \tag{A.16}$$

$$I_{-}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma_{b,k}(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_2\Psi_2\left[\begin{matrix}(\gamma_1-\chi+1,1) & (\gamma_2-\chi+1,1)\\ (1-\chi,1) & ((\gamma_1+\gamma_2+\delta-\chi+1,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{A.17}$$

$$I_{-}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma_b(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_2\Psi_2\left[\begin{matrix}(\gamma_1-\chi+1,1) & (\gamma_2-\chi+1,1)\\ (1-\chi,1) & ((\gamma_1+\gamma_2+\delta-\chi+1,1)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{A.18}$$

$$I_{-}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma_k(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_2\Psi_2\left[\begin{matrix}(\gamma_1-\chi+1,1) & (\gamma_2-\chi+1,1)\\ (1-\chi,1) & ((\gamma_1+\gamma_2+\delta-\chi+1,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{A.19}$$

$$I_{-}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_2\Psi_2\left[\begin{matrix}(\gamma_1-\chi+1,1) & (\gamma_2-\chi+1,1)\\ (1-\chi,1) & ((\gamma_1+\gamma_2+\delta-\chi+1,1)\end{matrix}\middle| -e^{\xi}\right]. \tag{A.20}$$

If $\gamma,\delta\in\mathbb{C}$; $\Re(\chi+p)<1+\Re(\gamma)$, then the E-K fractional transform of multiplicity $m=1$ or the right handed Erdélyi–Kober integral operator containing the Laplace transform of the extended k-gamma function and its special cases are given by

$$I_{0+}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_{b,k}(z);s\} = 2\pi s^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\ (\chi+\gamma+\delta,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{A.21}$$

$$I_{0+}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_b(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\ (\chi+\gamma+\delta,1)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{A.22}$$

$$I_{0+}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_k(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\ (\chi+\gamma+\delta,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{A.23}$$

$$I_{0+}^{\gamma,\delta} s^{\chi-1} L\{\Gamma(z);s\} = 2\pi s^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\ (\chi+\gamma+\delta,1)\end{matrix}\middle| -e^{\xi}\right], \tag{A.24}$$

$$I_{0-}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_{b,k}(z);s\} = 2\pi s^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\ (\gamma+\delta-\chi+1,-1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{A.25}$$

$$I_{0-}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_b(z);s\} = 2\pi s^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\ (\gamma+\delta-\chi+1,-1)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{A.26}$$

$$I_{0-}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_k(z);s\} = 2\pi s^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\ (\gamma+\delta-\chi+1,-1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{A.27}$$

$$I_{0-}^{\gamma,\delta} s^{\chi-1} L\{\Gamma(z);s\} = 2\pi s^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\ (\gamma+\delta-\chi+1,-1)\end{matrix}\middle| -e^{\xi}\right]. \tag{A.28}$$

If $\gamma,\delta\in\mathbb{C}$; $\Re(\chi+p)<1+\Re(\gamma)$, then the E-K fractional transform of multiplicity $m=1$ or the right handed Erdélyi–Kober integral operator involving the Laplace transform of the extended k-gamma function and its special cases are given by

$$I_{0+}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_{b,k}(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\ (\chi+\gamma+\delta,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{A.29}$$

$$I_{0+}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_b(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\ (\chi+\gamma+\delta,1)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{A.30}$$

$$I_{0+}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_k(z);s\} = 2\pi s^{\chi-\gamma_1-1} {}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\ (\chi+\gamma+\delta,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{A.31}$$

$$I_{0+}^{\gamma,\delta} s^{\chi-1} L\{\Gamma(z);s\} = 2\pi s^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\chi+\gamma,1)\\ (\chi+\gamma+\delta,1)\end{matrix}\middle| -e^{\xi}\right], \tag{A.32}$$

$$I_{0-}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_{b,k}(z);s\} = 2\pi s^{\chi-1} {}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\ (\gamma+\delta-\chi+1,-1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{A.33}$$

$$I_{0-}^{\gamma,\delta}s^{\chi-1}L\{\Gamma_b(z);s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\(\gamma+\delta-\chi+1,-1)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{A.34}$$

$$I_{0-}^{\gamma,\delta}s^{\chi-1}L\{\Gamma_k(z);s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\(\gamma+\delta-\chi+1,-1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{A.35}$$

$$I_{0-}^{\gamma,\delta}s^{\chi-1}L\{\Gamma(z);s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\gamma-\chi+1,-1)\\(\gamma+\delta-\chi+1,-1)\end{matrix}\middle| -e^{\xi}\right]. \tag{A.36}$$

If $\delta, \chi \in \mathbb{C}, \Re(\delta) > 0, \Re(\chi) > 0$, then the E-K fractional transform of multiplicity $m = 1$ or the left handed Riemann-Liouville (R-L) integral operator involving the extended k-gamma function and its special cases are given by

$$I_{0+}^{\delta}s^{\chi-1}L\{\Gamma_{b,k}(z);s\} = 2\pi s^{\chi-\gamma_1-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\delta+\chi,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{A.37}$$

$$I_{0+}^{\delta}s^{\chi-1}L\{\Gamma_b(z);s\} = 2\pi s^{\chi-\gamma_1-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\delta+\chi,1)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{A.38}$$

$$I_{0+}^{\delta}s^{\chi-1}L\{\Gamma_k(z);s\} = 2\pi s^{\chi-\gamma_1-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\delta+\chi,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{A.39}$$

$$I_{0+}^{\delta}s^{\chi-1}L\{\Gamma(z);s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\(\delta+\chi,1)\end{matrix}\middle| -e^{\xi}\right], \tag{A.40}$$

$$I_{-}^{\delta}s^{\chi-1}L\{\Gamma_{b,k}(z);s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(1-\delta-\chi,-1)\\(1-\chi,-1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{A.41}$$

$$I_{-}^{\delta}s^{\chi-1}L\{\Gamma_b(z);s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(1-\delta-\chi,-1)\\(1-\chi,-1)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{A.42}$$

$$I_{-}^{\delta}s^{\chi-1}L\{\Gamma_k(z);s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(1-\delta-\chi,-1)\\(1-\chi,-1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{A.43}$$

$$I_{-}^{\delta}s^{\chi-1}L\{\Gamma(z);s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(1-\delta-\chi,-1)\\(1-\chi,-1)\end{matrix}\middle| -e^{\xi}\right]. \tag{A.44}$$

B. Generalized Fractional Derivatives Involving the Laplace Transform of the Extended k-Gamma Function

The E-K fractional derivatives of multiplicity m containing the Laplace transform of the extended k-gamma *function* is given by

$$\left(D_{(\beta k),m}^{(\gamma k)_1^m,(\delta k)}s^{\chi-1}L\{\Gamma_{b,k}(z);s\}\right) = 2\pi s^{\chi-1}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_k+\delta_k+1+\dfrac{c}{\beta_k},\dfrac{1}{\beta_k}\right)_1^m\\\left(\gamma_k+1+\dfrac{c}{\beta_k},\dfrac{1}{\beta_k}\right)_1^m\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{B.1}$$

$$\left(D_{(\beta k),m}^{(\gamma k)_1^m,(\delta k)}s^{\chi-1}L\{\Gamma_{b}(z);s\}\right) = 2\pi s^{\chi-1}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_k+\delta_k+1+\dfrac{c}{\beta_k},\dfrac{1}{\beta_k}\right)_1^m\\\left(\gamma_k+1+\dfrac{c}{\beta_k},\dfrac{1}{\beta_k}\right)_1^m\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{B.2}$$

$$\left(D_{(\beta k),m}^{(\gamma k)_1^m,(\delta k)}s^{\chi-1}L\{\Gamma_{k}(z);s\}\right) = 2\pi s^{\chi-1}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_k+\delta_k+1+\dfrac{c}{\beta_k},\dfrac{1}{\beta_k}\right)_1^m\\\left(\gamma_k+1+\dfrac{c}{\beta_k},\dfrac{1}{\beta_k}\right)_1^m\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{B.3}$$

$$\left(D_{(\beta k),m}^{(\gamma k)_1^m,(\delta k)}s^{\chi-1}L\{\Gamma(z);s\}\right) = 2\pi s^{\chi-1}{}_m\Psi_m\left[\begin{matrix}\left(\gamma_k+\delta_k+1+\dfrac{c}{\beta_k},\dfrac{1}{\beta_k}\right)_1^m\\\left(\gamma_k+1+\dfrac{c}{\beta_k},\dfrac{1}{\beta_k}\right)_1^m\end{matrix}\middle| -e^{\xi}\right]. \tag{B.4}$$

The multiple E-K fractional derivatives with $m = 3$ or the Marichev–Saigo–Maeda fractional derivative operator involving the Laplace transform of the extended k-gamma *function* is given by

$$D_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma_{b,k}(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi-\gamma_2+\gamma_1,1) & \left(\chi+\gamma_{1+}\gamma_1'+\gamma_2'-\delta,1\right)\\ (\chi-\gamma_2,1) & \left(\chi-\delta+\gamma_1++\gamma_2',1\right) & \left(\chi-\delta+\gamma_1'+\gamma_1,1\right)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{B.5}$$

$$D_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma_{b}(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi-\gamma_2+\gamma_1,1) & \left(\chi+\gamma_{1+}\gamma_1'+\gamma_2'-\delta,1\right)\\ (\chi-\gamma_2,1) & \left(\chi-\delta+\gamma_1++\gamma_2',1\right) & \left(\chi-\delta+\gamma_1'+\gamma_1,1\right)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{B.6}$$

$$D_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma_{k}(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi-\gamma_2+\gamma_1,1) & \left(\chi+\gamma_{1+}\gamma_1'+\gamma_2'-\delta,1\right)\\ (\chi-\gamma_2,1) & \left(\chi-\delta+\gamma_1++\gamma_2',1\right) & \left(\chi-\delta+\gamma_1'+\gamma_1,1\right)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{B.7}$$

$$D_{0+}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}(\chi,1) & (\chi-\gamma_2+\gamma_1,1) & \left(\chi+\gamma_{1+}\gamma_1'+\gamma_2'-\delta,1\right)\\ (\chi-\gamma_2,1) & \left(\chi-\delta+\gamma_1++\gamma_2',1\right) & \left(\chi-\delta+\gamma_1'+\gamma_1,1\right)\end{matrix}\middle| -e^{\xi}\right], \tag{B.8}$$

$$D_{-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma_{b,k}(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}\left(1-\chi+\gamma_2',1\right) & \left(1+\gamma_2'-\chi-\gamma_2+\gamma_1,1\right) & \left(1-\chi-\gamma_{1-}\gamma_1'+\delta,1\right)\\ (1-\chi,1) & \left(1-\chi-\gamma_1'+\gamma_2',1\right) & \left(1-\chi+\delta-\gamma_1'-\gamma_1-\gamma_2,1\right)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{B.9}$$

$$D_{-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma_{b}(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}\left(1-\chi+\gamma_2',1\right) & \left(1+\gamma_2'-\chi-\gamma_2+\gamma_1,1\right) & \left(1-\chi-\gamma_{1-}\gamma_1'+\delta,1\right)\\ (1-\chi,1) & \left(1-\chi-\gamma_1'+\gamma_2',1\right) & \left(1-\chi+\delta-\gamma_1'-\gamma_1-\gamma_2,1\right)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{B.10}$$

$$D_{-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma_{k}(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}\left(1-\chi+\gamma_2',1\right) & \left(1+\gamma_2'-\chi-\gamma_2+\gamma_1,1\right) & \left(1-\chi-\gamma_{1-}\gamma_1'+\delta,1\right)\\ (1-\chi,1) & \left(1-\chi-\gamma_1'+\gamma_2',1\right) & \left(1-\chi+\delta-\gamma_1'-\gamma_1-\gamma_2,1\right)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{B.11}$$

$$D_{-}^{\gamma_1,\gamma_1',\gamma_2,\gamma_2',\delta} s^{\chi-1} L\{\Gamma(z);s\} = 2\pi s^{\delta+\chi-\gamma_1-\gamma_1'-1} {}_3\Psi_3\left[\begin{matrix}\left(1-\chi+\gamma_2',1\right) & \left(1+\gamma_2'-\chi-\gamma_2+\gamma_1,1\right) & \left(1-\chi-\gamma_{1-}\gamma_1'+\delta,1\right)\\ (1-\chi,1) & \left(1-\chi-\gamma_1'+\gamma_2',1\right) & \left(1-\chi+\delta-\gamma_1'-\gamma_1-\gamma_2,1\right)\end{matrix}\middle| -e^{\xi}\right]. \tag{B.12}$$

The multiple E-K fractional derivatives with $m = 2$ or the Saigo fractional derivatives operator involving the Laplace transform of the extended k-gamma *function* is given by

$$\begin{aligned}&D_{0+}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma_{b,k}(z);s\}\\ &= 2\pi s^{\chi-\gamma_1-1} {}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\delta+\gamma_2+\gamma_1,1)\\ (\chi+\gamma_2,1) & (\chi+\delta,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right],\end{aligned} \tag{B.13}$$

$$\begin{aligned} &D_{0+}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma_b(z)\,;s\} \\ &= 2\pi s^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\delta+\gamma_2+\gamma_1,1)\\ (\chi+\gamma_2,1) & (\chi+\delta,1)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \end{aligned} \tag{B.14}$$

$$\begin{aligned} &D_{0+}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma_k(z)\,;s\} \\ &= 2\pi s^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\delta+\gamma_2+\gamma_1,1)\\ (\chi+\gamma_2,1) & (\chi+\delta,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \end{aligned} \tag{B.15}$$

$$\begin{aligned} &D_{0+}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma(z)\,;s\} \\ &= 2\pi s^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\chi,1) & (\chi+\delta+\gamma_2+\gamma_1,1)\\ (\chi+\gamma_2,1) & (\chi+\delta,1)\end{matrix}\middle| -e^{\xi}\right], \end{aligned} \tag{B.16}$$

$$\begin{aligned} &D_{-}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma_{b,k}(z)\,;s\} \\ &= 2\pi s^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(1-\chi-\gamma_2,1) & (1-\chi+\delta+\gamma_1,1)\\ (1-\chi+\delta-\gamma_2,1) & (1-\chi,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \end{aligned} \tag{B.17}$$

$$\begin{aligned} &D_{-}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma_b(z)\,;s\} \\ &= 2\pi s^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(1-\chi-\gamma_2,1) & (1-\chi+\delta+\gamma_1,1)\\ (1-\chi+\delta-\gamma_2,1) & (1-\chi,1)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \end{aligned} \tag{B.18}$$

$$\begin{aligned} &D_{-}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma_k(z)\,;s\} \\ &= 2\pi s^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(1-\chi-\gamma_2,1) & (1-\chi+\delta+\gamma_1,1)\\ (1-\chi+\delta-\gamma_2,1) & (1-\chi,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \end{aligned} \tag{B.19}$$

$$\begin{aligned} &D_{-}^{\gamma_1,\gamma_2,\delta} s^{\chi-1} L\{\Gamma(z)\,;s\} \\ &= 2\pi s^{\chi-\gamma_1-1}{}_2\Psi_2\left[\begin{matrix}(\gamma_1-\chi+1,1) & (\gamma_2-\chi+1,1)\\ (1-\chi,1) & ((\gamma_1+\gamma_2+\delta-\chi+1,1)\end{matrix}\middle| -e^{\xi}\right]. \end{aligned} \tag{B.20}$$

If $\gamma, \delta \in \mathbb{C}\,; \mathfrak{R}(\chi+\mathrm{p}) < 1+\mathfrak{R}(\gamma)$, then the multiple E-K fractional derivatives with $m=1$ or the right handed Erdélyi–Kober derivatives operator involving the Laplace transform of the extended k-gamma function and its special cases are given by

$$\begin{aligned} &D_{0+}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_{b,k}(z)\,;s\} \\ &= 2\pi s^{\chi-\gamma_1-1}{}_1\Psi_1\left[\begin{matrix}(\gamma+\delta+\chi,1)\\ (\gamma+\chi,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \end{aligned} \tag{B.21}$$

$$\begin{aligned} &D_{0+}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_b(z)\,;s\} \\ &= 2\pi s^{\chi-\gamma_1-1}{}_1\Psi_1\left[\begin{matrix}(\gamma+\delta+\chi,1)\\ (\gamma+\chi,1)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \end{aligned} \tag{B.22}$$

$$D_{0+}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_k(z)\,;s\} = 2\pi s^{\chi-\gamma_1-1}{}_1\Psi_1\left[\begin{matrix}(\gamma+\delta+\chi,1)\\ (\gamma+\chi,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{B.23}$$

$$D_{0+}^{\gamma,\delta} s^{\chi-1} L\{\Gamma(z)\,;s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\gamma+\delta+\chi,1)\\ (\gamma+\chi,1)\end{matrix}\middle| -e^{\xi}\right], \tag{B.24}$$

$$\begin{aligned} &D_{-}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_{b,k}(z)\,;s\} \\ &= 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(1-\chi+\gamma+\delta,1)\\ (1-\chi+\gamma,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \end{aligned} \tag{B.25}$$

$$D_{-}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_b(z)\,;s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(1-\chi+\gamma+\delta,1)\\ (1-\chi+\gamma,1)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{B.26}$$

$$D_{-}^{\gamma,\delta} s^{\chi-1} L\{\Gamma_k(z)\,;s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(1-\chi+\gamma+\delta,1)\\ (1-\chi+\gamma,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{B.27}$$

$$D_{-}^{\gamma,\delta} s^{\chi-1} L\{\Gamma(z)\,;s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(1-\chi+\gamma+\delta,1)\\ (1-\chi+\gamma,1)\end{matrix}\middle| -e^{\xi}\right]. \tag{B.28}$$

If $\delta, \chi \in \mathbb{C}, \mathfrak{R}(\delta) > 0, \mathfrak{R}(\chi) > 0$,, then the multiple E-K fractional derivatives with $m=1$ or the left and right handed Riemann-Liouville (R-L) fractional derivatives involving the Laplace transform of the extended k-gamma function and its special cases are given by

$$D_{0+}^{\delta} s^{\chi-1} L\{\Gamma_{b,k}(z)\,;s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\ (\chi-\delta,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{B.29}$$

$$D_{0+}^{\delta} s^{\chi-1} L\{\Gamma_{b,k}(z)\,;s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\ (\chi-\delta,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}-\frac{b^k e^{-k\xi}}{k}\right], \tag{B.30}$$

$$D_{0+}^{\delta} s^{\chi-1} L\{\Gamma_b(z)\,;s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\ (\chi-\delta,1)\end{matrix}\middle| -e^{\xi}-be^{-\xi}\right], \tag{B.31}$$

$$D_{0+}^{\delta} s^{\chi-1} L\{\Gamma_k(z)\,;s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\ (\chi-\delta,1)\end{matrix}\middle| -\frac{e^{k\xi}}{k}\right], \tag{B.32}$$

$$D_{0+}^{\delta} s^{\chi-1} L\{\Gamma(z)\,;s\} = 2\pi s^{\chi-1}{}_1\Psi_1\left[\begin{matrix}(\chi,1)\\ (\chi-\delta,1)\end{matrix}\middle| -e^{\xi}\right], \tag{B.33}$$

$$D_{-}^{\delta}s^{\chi-1}L\{\Gamma_{b,k}(z)\,;s\}=2\pi s^{\chi-1}{}_{1}\Psi_{1}\left[\begin{matrix}(\delta-\chi+1,1)\\(1-\chi,1)\end{matrix}\middle|-\frac{e^{k\xi}}{k}-\frac{b^{k}e^{-k\xi}}{k}\right],\tag{B.34}$$

$$D_{-}^{\delta}s^{\chi-1}L\{\Gamma_{b}(z)\,;s\}=2\pi s^{\chi-1}{}_{1}\Psi_{1}\left[\begin{matrix}(\delta-\chi+1,1)\\(1-\chi,1)\end{matrix}\middle|-e^{\xi}-be^{-\xi}\right],\tag{B.35}$$

$$D_{-}^{\delta}s^{\chi-1}L\{\Gamma_{k}(z)\,;s\}=2\pi s^{\chi-1}{}_{1}\Psi_{1}\left[\begin{matrix}(\delta-\chi+1,1)\\(1-\chi,1)\end{matrix}\middle|-\frac{e^{k\xi}}{k}\right],\tag{B.36}$$

$$D_{-}^{\delta}s^{\chi-1}L\{\Gamma(z)\,;s\}=2\pi s^{\chi-1}{}_{1}\Psi_{1}\left[\begin{matrix}(\delta-\chi+1,1)\\(1-\chi,1)\end{matrix}\middle|-e^{\xi}\right].\tag{B.37}$$

Data Availability

The study did not report any data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally.

Acknowledgments

The author (AT) is also thankful to the deanship of scientific research at Majmaah University for providing excellent research facilities.

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