Hindawi
Advances in Mathematical Physics
Volume 2022, Article ID 4345342, 9 pages
https://doi.org/10.1155/2022/4345342

Research Article

Shape Invariance of Solvable Schrödinger Equations with the Generalized Hyperbolic Pöschl-Teller Potential

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Received 14 April 2022; Revised 18 June 2022; Accepted 23 June 2022; Published 21 July 2022

Academic Editor: S. A. Edalatpanah Edalatpanah

In atomic and molecular physics, the Pöschl-Teller potential and its modified form (hyperbolic Pöschl-Teller potential) are particularly significant potentials. It is of great importance to study the Schrödinger equation with those potentials. In this paper, we further extend the hyperbolic Pöschl-Teller potential through generalizing the superpotential of that potential of the form $A\tanh(\alpha x)$-$B\coth(\alpha x)$ to the more general form -$A\tanh(npx)$-$B\coth(mpx)$. First, we introduce briefly the shape invariance and the potential algebra in supersymmetric quantum mechanics. Second, we derive three additive shape invariances, which are related to parameters A and B of the partner potentials with the generalized superpotential, and discuss the eigenfunctions and eigenvalues in detail. Although the superpotential has two parameters, those shape invariances still belong to the one-parameter form. The reason is that there is always a constraint relationship between A and B in the additive shape invariance of the partner potentials. Third, through the potential algebra approach, we obtain the relevant shape invariance and calculate the corresponding eigenvalue of the Schrödinger equation with the potential of the generalized superpotential. The calculation shows that the algebraic form shape invariance of the partner potentials with that superpotential is anastomotic to the above. Last, we make a summary and outlook.

1. Introduction

Although quantum mechanics has received brilliant achievements in physics, it still faces several unsolved inconsistencies [1]. Matrix mechanics and wave mechanics are two traditional methods for solving the Schrödinger equation [2]. However, both methods are difficult to use and are unsuitable for observing physical images. Therefore, it is very important to find a new method of solving the Schrödinger equation. Supersymmetric quantum mechanics (SUSYQM), developed based on supersymmetric field theory, provides a new method for solving the Schrödinger equation [3–6]. Supersymmetric field theory unifies the anticommutation and commutation relations between operators into a closed algebra and links the fermions and bosons by specific transformation [7]. To avoid the degeneration of the fermion spectrum and boson spectrum, the system must have supersymmetric spontaneous breaking. Witten proposed the quantum mechanical supersymmetric “toy” model to solve the breaking problem in field theory [7]. In this model, the Schrödinger equation with exactly solvable potential is not only exactly solvable, but also the physical image is clear. Therefore, SUSYQM has developed rapidly. Further research also shows that the factorization method of solvable models can incorporate into the theoretical framework of SUSYQM, and can be widely used in atomic, molecular physics, and condensed matter physics. Therefore, it is of great significance to further study and expand the solvable potentials based on SUSYQM.

Shape invariance study gives us a powerful method to solve the Schrödinger equation more easily [8]. The shape invariance of the partner potential can determine the energy spectrum and eigenfunction of Hamiltonian without solving the Schrödinger equation. In addition, the potential algebraic approach also is a powerful tool in SUSYQM. Potential algebra can also help us get the energy spectrum and eigenfunction of Hamiltonian, even obtain the scattering amplitude and spectrum [9]. In the 7.2 section of the reference [10], we can learn that the two approaches are essentially equivalent.

In atomic and molecular physics, the trigonometric Pöschl-Teller potential and its modified form (hyperbolic Pöschl-Teller) are significant potentials and widely used [11, 12]. Further generalizing the hyperbolic Pöschl-Teller potential is undoubtedly essential. In recent years, many scholars have made extensive studies on the hyperbolic potential from different perspectives [12–14]. Although they have studied based on SUSYQM, they only give a form of shape invariance. The reason is that they do not notice that this potential has two-parameter characteristics.

In this paper, based on SUSYQM, we study the shape invariance of the Schrödinger equation with the generalized hyperbolic Pöschl-Teller potential. In Section 2, we briefly introduce SUSYQM. In Section 3, according to the superpotential of the hyperbolic Pöschl-Teller potential (Pöschl-Teller II potential), our group makes a more general extension (to generalize the superpotential of that potential of the form $A\tanh(\alpha x)$-$B\coth(\alpha x)$ [15] to the form -$A\tanh(npx)$-$B\coth(mpx)$) and focus on the parameters A and B. Then, we derive three additive shape invariances of partner potentials with the generalized potential. Further, we research the coefficient-dependent eigenfunctions and eigenvalues in terms of the various value ranges of A and B deeply. In Section 4, we use the potential algebra method, and obtain the shape invariances of potential algebraic forms, which are anastomotic to those of the partner potentials. In the last section, we make a summary and outlook.

2. Supersymmetric Quantum Mechanics

In SUSYQM, a superpotential $W(x,a)$ can generate two partner potentials

$$V_{\pm}(x,a) = W^2(x,a) \pm \frac{\hbar}{\sqrt{2m}}\frac{dW(x,a)}{dx}, \tag{1}$$

and the partner Hamiltonians

$$H_{\pm} = -\frac{\hbar^2}{2m}\frac{d^2}{dx} + V_{\pm}(x,a). \tag{2}$$

The partner Hamiltonians have identical spectra for an arbitrary function $W(x,a)$. The only exceptional case may be that one of the two partner Hamiltonians has a zero-energy bound state [7, 9, 10]. For brevity, we set $\hbar = 2m = 1$. According to superpotential $W(x,a)$, we can also define the increasing and decreasing operators A^+ and A^-:

$$A^{\pm}(x,a) = \mp\frac{d}{dx} + W(x,a), \tag{3}$$

and the partner Hamiltonians are given with A^+ and A^+:

$$H_- = A^+A^-, H_+ = A^-A^+. \tag{4}$$

If we suppose the $\psi_0^-(x,a)$ is a normalized function, the operator's eigenvalues E_n^- are equal to or greater than zero,

$$E_n^- \geq 0. \tag{5}$$

Therefore, the operator H_- is called the "semi-positive definite" operator [10]. It is widely known that the partner Hamiltonians' eigenfunctions are related by the increasing and decreasing operators. All excited states ψ_n^- satisfy equations: $\psi_{n-1}^+ = A^-\psi_n^-$, and the corresponding eigenvalues meet $E_{n-1}^+ = E_n^-$ $(n = 1, 2, 3 \cdots\cdots)$. If the ground state of H_- has nothing to do with any state of H_+ and meets $A^-\psi_0^- = 0$ which has a normalized solution, then the system is supersymmetric. Otherwise, it is broken [10].

If SUSY is unbroken, it has a normalized ground state wave function for the potential V_- [16]. According to the superpotential, we can obtain the zero-energy ground state wave function $\psi_0^-(x,a)$:

$$\psi_0^-(x,a) = N\exp\left(-\int_{x_0}^{x} W(x,a)dx\right), \tag{6}$$

where N is a normalized constant. And the ground energy eigenvalue meet $E_0^- = 0$. Besides the ground state, all its energy eigenvalues of the potential V_- are obtained with those of the partner potential V_+ [17], i.e.,

$$E_n^+ = E_{n+1}^-. \tag{7}$$

Moreover, the relationships among the eigenstates are

$$\psi_n^+ = \frac{1}{\sqrt{(E_{n+1}^-)}}A^-\psi_{n+1}^-, \psi_n^- = \frac{1}{\sqrt{(E_{n-1}^+)}}A^+\psi_{n-1}^+. \tag{8}$$

The Hamiltonians $H_{\pm}$ and the partner potentials $V_{\pm}(x,a)$ satisfy the shape invariance [18] which have the following algebraic structure:

$$\begin{aligned} H_+(x,a_0) + g(a_0) &= H_-(x,a_1) + g(a_1), \\ V_+(x,a_0) + g(a_0) &= V_-(x,a_1) + g(a_1). \end{aligned} \tag{9}$$

In the equations above, a_1 is an additive function of a_0: $a_1 = f(a_0)$ which is independent of x. The eigenvalues meet

$$E_n^+(a_0) + g(a_0) = E_n^-(a_1) + g(a_1), \tag{10}$$

and the eigenfunctions satisfy

$$\psi_n^+(x,a_0) = \psi_n^-(x,a_1). \tag{11}$$

Based on SUSYQM, isospectrality [10], and utilizing Equation (11), we keep iterating Equation (10) and figure up

$$E_n^-(a_n) = g(a_n) - g(a_0). \tag{12}$$

Analogously, taking no account of the normalization constant temporarily, we obtain:

$$\psi_n^-(x,a_0) \propto A^+(x,a_0)A^+(x,a_1)\cdots A^+(x,a_{n-1})\psi_0^-(x,a_n). \tag{13}$$

Then, we can also calculate the energy eigenvalues.

3. The Shape Invariance with the Generalized Hyperbolic Pöschl-Teller Potential of the Superpotential -Atanh(npx)-Bcoth(mpx)

In this part, by appropriate redefinition of the parameters of superpotential ($A \tanh(\alpha x) - B \coth(\alpha x)(A > B > 0)$) of the hyperbolic Pöschl-Teller potential, we obtain the generalized hyperbolic Pöschl-Teller superpotential:

$$W(x, A, B) = -A \tanh(npx) - B \coth(mpx)(m \neq n). \quad (14)$$

where A and B are two parameters (the following research is carried out around these two parameters), p is an arbitrary positive real number, and m and n are arbitrary positive integers and are not equal to each other. The superpotential $W(x, A, B)$ is shown in Figure 1. As can be seen from the figure, the change has not taken place in curvilinear trends when p is taken at different values, and only the curve moves up as the value gets greater. To observe the image in more detail, by changing the range of x, in Figure 2, we draw the curve of superpotential with the different values of A and B.

According to $V_{\pm}(x, a) = W^2(x, a) \pm dW(x, a)/dx$, we can get

$$\begin{cases} V_-(x, A_1, B_1) = -A_1(A_1 - np)\,\mathrm{sech}^2(npx) + B_1(B_1 - mp)\,\mathrm{csch}^2(mpx) + 2A_1B_1 \tanh(npx) \coth(mpx) + {A_1}^2 + {B_1}^2 \\ V_+ = (x, A_0, B_0) = -A_0(A_0 + np)\,\mathrm{sech}^2(npx) + B_0(B_0 + mp)\,\mathrm{csch}^2(mpx) + 2A_0B_0 \tanh(npx) \coth(mpx) + {A_0}^2 + {B_0}^2 \end{cases}. \quad (15)$$

$V_{\pm}(x, A, B)$ meet the shape invariance:

$$V_+(x, A_0, B_0) = V_-(x, A_1, B_1) + g(A_1, B_1) - g(A_0, B_0). \quad (16)$$

where the parameters A_1 and B_1, respectively, are the functions of the parameters A_0 and B_0 ($A_1 = f(A_0), B_1 = h(B_0)$). $R(A_0, B_0)$ is a function of the parameters A_0 and B_0 ($R(A_0, B_0) = g(A_1, B_1) - g(A_0, B_0)$). The partner potentials $V_{\pm}(x, A_0, B_0)$ are shown in Figure 3.

There is also a corresponding relationship for $H^{\pm}(x, A, B)$ and $E_k^+(A, B)$:

$$\begin{aligned} H^+(x, A_0, B_0) + g(A_0, B_0) &= H^-(x, A_1, B_1) + g(A_1, B_1), \\ E_k^+(A_0, B_0) + g(A_0, B_0) &= E_k^-(A_1, B_1) + g(A_1, B_1). \end{aligned} \quad (17)$$

And there are

$$\psi_k^+(x, A_0, B_0) = \psi_k^-(x, A_1, B_1), \quad (18)$$

$$E_k^-(A_0, B_0) = g(A_k, B_k) - g(A_0, B_0) = {A_0}^2 + {B_0}^2 - \left({A_k}^2 + {B_k}^2\right). \quad (19)$$

The ground state of $H^-(x, A_0, B_0)$ is

$$\begin{aligned} \psi_0^-(x, A_0, B_0) &= N e^{-\int_{x_0}^{x} W(x, A_0, B_0)dx} \\ &= N(\cosh(npx))^{A_0/np}(\sinh(mpx))^{B_0/mp}. \end{aligned} \quad (20)$$

And we can obtain the first excited state by

$$\begin{aligned} \psi_1^-(x, A_0, B_0) &\sim A^+(x, A_0, B_0)\psi_0^-(x, A_1, B_1) \\ &\sim A^+(x, A_0, B_0)(\cosh(npx))^{A_1/np}(\sinh(mpx))^{B_1/mp}. \end{aligned} \quad (21)$$

According to Equation (18), there is

$$\begin{aligned} \psi_k^-(x, A_0, B_0) \sim{}& A^+(x, A_0, B_0)A^+(x, A_1, B_1)A^+(x, A_2, B_2) \cdots A^+ \\ &\cdot (x, A_{k-1}, B_{k-1})\psi_0^-(x, A_k, B_k). \end{aligned} \quad (22)$$

To get the shape invariance of Equation (16), the coefficients of those terms that contain the independent variable x must cancel each other in Equation (15), i.e.,

$$A_0(A_0 + np) = A_1(A_1 - np), B_0(B_0 + mp) = B_1(B_1 - mp), \quad (23)$$

$$A_0B_0 = A_1B_1. \quad (24)$$

Through solving Equation (23), we obtain

$$\begin{cases} A_1 = -A_0 \text{or} A_1 = A_0 + np \\ B_1 = -B_0 \text{or} B_1 = B_0 + \mathrm{m}p \end{cases}. \quad (25)$$

Therefore, we acquire four kinds of solutions: (I) $A_1 = A_0 + np, \mathrm{B}_1 = B_0 + \mathrm{m}p$; (II) $A_1 = A_0 + np, B_1 = -B_0$; (III) $A_1 = -A_0, \mathrm{B}_1 = B_0 + mp$; and (IV) $A_1 = -A_0, B_1 = -B_0$. (Because this situation (IV) does not meet the additive feature, we will not discuss it here.)

3.1. Case I: $A_1 = A_0 + np, B_1 = B_0 + mp$. Substituting $A_1 = A_0 + np, B_1 = B_0 + mp$ into Equation (24), we have

$$\frac{B_0}{mp} + 1 = \frac{-A_0}{np}, \quad (26)$$

and Equation (19) is rewritten as

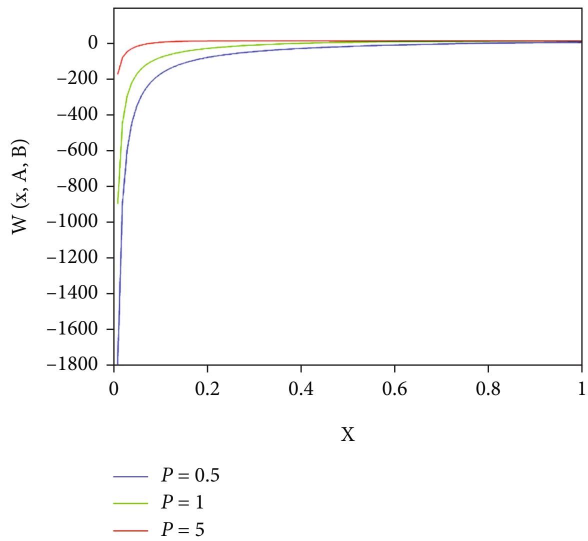

Figure 1: Superpotential $W(x, A, B)$ $(m = 2, n = 3, A = -28.5, B = 18)$.

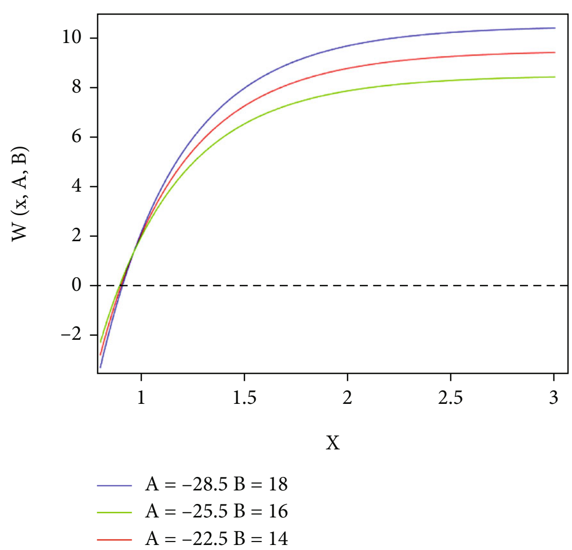

Figure 2: Superpotential $W(x, A, B)$ $(m = 2, n = 3, p = 0.5)$.

$$E_k^-(A_0, B_0) = -2A_0knp - 2B_0kmp - k^2n^2p^2 - k^2m^2p^2. \quad (27)$$

(The k is an integer greater than zero.)

Since $E_k^- \geq 0$, the values of parameters in Equation (27) are limited. Next, we will discuss the value situation of each parameter, specifically from the value of A_0 and B_0.

(i) $A_0 > 0, B_0 > 0$.

Since the parameters meet $k \geq 0, n > 0, p > 0, m > 0$ in Equation (27), there is

$$E_k^-(A_0, B_0) = 0, \quad (28)$$

only when $k = 0, E_k^- \geq 0$.

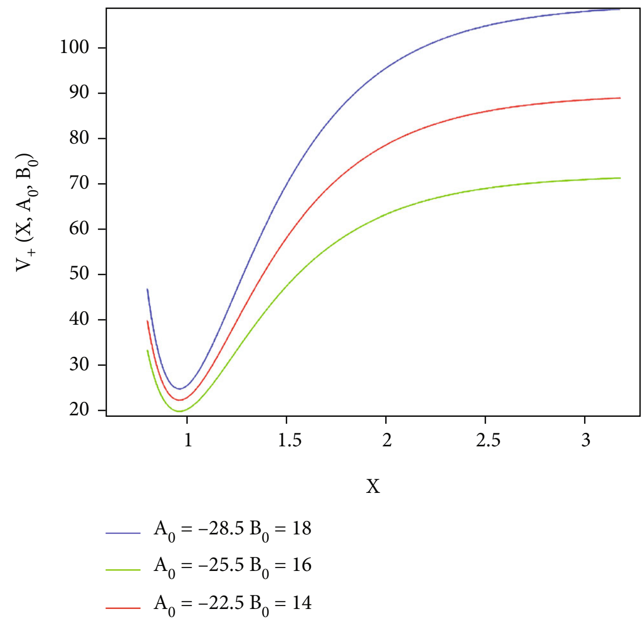

Figure 3: Partner Potentials $V_+ = (x, A_0, B_0)$ $(m = 2, n = 3, p = 0.5)$.

(ii) $A_0 < 0, B_0 < 0$.

According to $A_0 < 0, B_0 < 0$, the range of k is solved:

$$k < \frac{-A_0}{np} = 1 + \frac{B_0}{mp}, k < \frac{-B_0}{mp}. \quad (29)$$

To further determine the precise range of k, we must compare $1 + (B_0/mp)$ with $-B_0/mp$. If $(-B_0/mp) \geq 1 + (B_0/mp)$, we not only determine the range of k: $k < 1 + (B_0/mp)$, but also know $-B_0 \geq (mp/2)$. Then, the precise data range of k: $0 \leq k < (1/2)$ is confirmed. The unique value can be chosen: $k = 0$, and we figure out $E_0^- = 0$. If $(-B_0/mp) < 1 + (B_0/mp)$, we can get similar inequations: $k < (-B_0/mp)$, $-B_0 < (mp/2)$ and we have $0 \leq k < (1/2)$. So the k has sole value: $k = 0$, then we obtain $E_0^- = 0$.

(iii) $A_0 > 0, B_0 < 0$.

It is similar to Equation (29), we acquire the new range about k:

$$k > \frac{-A_0}{np} = 1 + \frac{B_0}{mp}, k < \frac{-B_0}{mp}. \quad (30)$$

Similarly, if we assume $(-B_0/mp) \leq 1 + (B_0/mp)$, by simplifying the above equation, the inequation about B_0 is acquired: $-B_0 \leq (mp/2)$. Further, we can deduce that $k > 1 + (B_0/mp) > (1/2)$, $k < (-B_0/mp) < (1/2)$, those which are contradictory to each other, that is to say, the desired value of k is nonexistent. If $(-B_0/mp) > 1 + (B_0/mp)$, we acquire the two inequations about parameters B_0 and k: $-B_0 > (mp/2)$, $1 + (B_0/mp) < k < (-B_0/mp)$. We need to compare the lower bound of k with zero. In the first situation, we have $1 + (B_0/mp) > 0 \Longrightarrow -mp < B_0 < 0$. When $0 < 1 + (B_0/mp) <$

1 and $(-B_0/mp) < 1$, we can get $1 + (B_0/mp) < k < (-B_0/mp)$ in which there is not an integer solution for k. While in another situation: $1 + (B_0/mp) < 0$, we can figure out that $B_0 < -mp, 0 \le k < (-B_0/mp)$. After confirming the range of parameters, we draw Figure 4 to observe the superpotential $W(x, A, B)$.

From Figure 4, we observe that W and x have a negative correlation, and it is similar to Figure 1 that the change of p makes no difference in the curvilinear trend. With the increase of the value of p, the image moves down as a whole.

To satisfy $E^-_{k+1} \ge E^-_k$, the value of k has to meet the condition:

$$k \le \frac{-(A_0 n + B_0 m)}{p(n^2 + m^2)} - \frac{1}{2}. \tag{31}$$

If $-(B_0/mp) \le (-(A_0 n + B_0 m)/p(n^2 + m^2)) - (1/2)$, there is $-B_0 \le (pm(n^2 - m^2)/4n^2)$. According to $B_0 < -mp$ and $0 \le k < (-B_0/mp)$, we can get the new range of k: $0 \le k < (-B_0/mp) \le (n^2 - m^2/4n^2) < (1/4)$, so there is the only result: $k = 0$, $E^-_0 = 0$. On the contrary, when $-(B_0/mp) \ge (-(A_0 n + B_0 m)/p(n^2 + m^2)) - (1/2)$, we can get the more precise range of B_0 and k: $-B_0 \ge (pm(n^2 - m^2)/4n^2)$, $0 \le k < (B_0/mp) - (m^2/n^2 + m^2) + (1/2) < (1/2)$. That is to say, $k = 0$, $E^-_0 = 0$ is eligible.

(iv) $A_0 < 0, B_0 > 0$.

In this case, there is $k < (-A_0/np) = 1 + (B_0/mp)$, $k > (-B_0/mp)$. On account of $B > 0$, we can get a sole reasonable situation:

$$\frac{-B_0}{mp} < 1 + \frac{B_0}{mp}, 0 \le k < 1 + \frac{B_0}{mp}. \tag{32}$$

Considering $E^-_{k+1} \ge E^-_k$, we can also acquire Equation (31). Similar to the discussion in case (iii), when $1 + (B_0/mp) < (-(A_0 n + B_0 m)/p(n^2 + m^2)) - (1/2)$, there is

$$\frac{B_0}{mp} < \frac{-n^2 - 3m^2}{4m^2} < 0, \tag{33}$$

which is inconsistent with our assumptions. If $1 + (B_0/mp) > (-(A_0 n + B_0 m)/p(n^2 + m^2)) - (1/2)$ is tenable, there is $(B_0/mp) > (-n^2 - 3m^2/4m^2)$, and $0 \le k < (-(A_0 n + B_0 m)/p(n^2 + m^2)) - (1/2)$. And we have

$$0 \le k < \frac{B_0(n^2 - m^2)}{mp(n^2 + m^2)} + \frac{n^2}{n^2 + m^2} - \frac{1}{2}. \tag{34}$$

In the Inequation(34), we demand $n^2 > m^2$. When the parameters $B_0, m, n,$ and p are confirmed, the range of k is entirely determined. According to Equation (26), the A_0 is confirmed. Then we can, respectively, obtain the following equation:

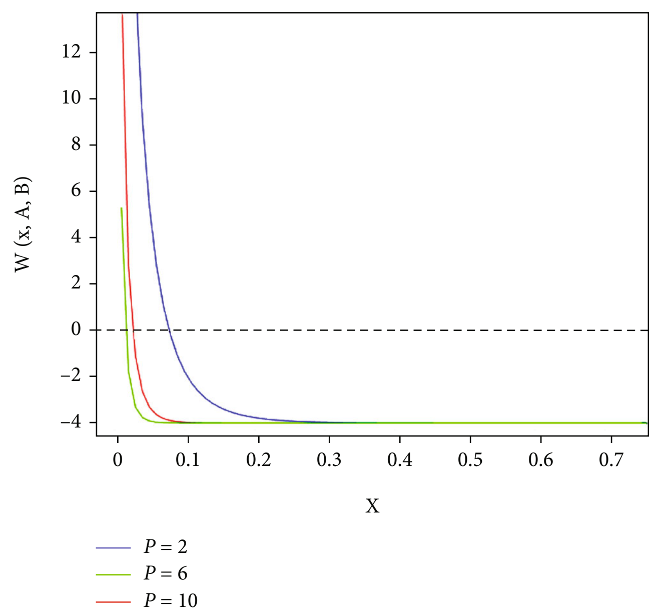

FIGURE 4: Superpotential $W(x, A, B)$ $(m = 5,\ n = 10,\ A = 10,\ B = -6)$.

$$\begin{aligned}
E^-_k(A_0) &= -2A_0 knp + \frac{2km^2 pA_0}{n} - k^2 n^2 p^2 - k^2 m^2 p^2 + 2km^2 p^2, \\
\psi^-_0(x, A_0) &\sim (\cosh\ (npx))^{A_0/\mathrm{np}} (\sinh\ (mpx))^{-(1 + (A_0/\mathrm{np}))}, \\
\psi^-_1(x, A_0) &\sim (\cosh\ (npx))^{A_0 + np/np} (\sinh\ (mpx))^{-A_0/np} \\
&\quad \cdot \left[(np - 2A_0) \tanh\ (npx) + \left(\frac{2A_0 m}{n} + mp \right) \coth\ (mpx) \right].
\end{aligned} \tag{35}$$

According to Equation (22), the other wave functions can also be calculated. At this point, we suppose the appropriate value for B_0, m, n, p: $B_0 = 18, m = 2, n = 3$ and $p = 0.5$. There is a maximum of k: $k = 7$ and $A_0 = -28.5$. In Table 1, we show the energy eigenvalues $E^-_k (k = 0, 1, \cdots, 7)$. In Figures 5 and 6, we show the ground state and first excited state wave functions.

3.2. Case II: $A_1 = A_0 + np$, $B_1 = -B_0$. We can get the following solutions through Equation (24):

$$B_0 = 0, A_1 = A_0 + np, \text{or } B_0 = -B_1 \ne 0, A_0 = -\frac{np}{2}. \tag{36}$$

For $B_0 = 0$, $A_1 = A_0 + np$, the superpotential is adapted as $W(x, A, B) = -A \tanh\ (npx)$ which can be treated as the special shape of Rosen-Morse II (hyperbolic) [10, 19]. There is a detailed discussion in Problem 4.2 of Ref. [11].

For the second solution in Equation (36), there are

$$\begin{aligned}
E^-_k(A_0) &= -knp(-np + knp), \\
\psi^-_0(x, A_0, B_0) &= (\cosh\ (npx))^{-1/2} (\sin\ (mpx))^{B_0/\mathrm{mp}}, \\
\psi^-_1(x, A_0, B_0) &\sim A^+(x, A_0, B_0) (\cosh\ (npx))^{-1/2} (\sin\ (mpx))^{B_0/mp} = 0.
\end{aligned} \tag{37}$$

Table 1: The energy eigenvalue E_k^-.

	$B_0 = 18, A_0 = -28.5, m = 2, n = 3, p = 0.5$							
k	0	1	2	3	4	5	6	7
E_k^-	0	46.25	86.00	119.25	146.00	166.25	180.00	187.25

To meet the supersymmetry of the system, the k only have two values: $k = 0, k = 1$. And the first excited state is missing. The eigenstate that satisfies the condition $-B_0 > (mp/2)$ and $1 + (B_0/mp) < k < (-B_0/mp)$ is found to be only a ground state:$k = 0, E_0^-(A_0) = 0$.

3.3. Case III: $A_1 = -A_0, B_1 = B_0 + mp$. It is similar to Case II, so we will not repreat the discussion. So far, we have systematically studied the shape invariances of the generalized hyperbolic Pöschl-Teller potential based on SUSYQM, and discussed in detail the influence of the shape invariance of parameters A and B, and analyzed the relationship between parameters and eigenvalues or eigenfunctions. The research results are summarized in Table 2.

4. Potential Algebra of the Shape Invariance with the Generalized Hyperbolic Pöschl-Teller Superpotential

In this section, the shape invariance is analyzed again by the potential algebra approach [20] with the generalized hyperbolic Pöschl-Teller superpotential of two parameters A and B. Let us make the following replacement:

$$\begin{aligned} & x \longrightarrow z, (A_0, B_0) \longrightarrow \chi\left(i\partial_{\phi A}, i\partial_{\phi B}\right) \\ & \Longrightarrow A^{\pm}(x, A_0, B_0) \longrightarrow \mathscr{A}^{\pm}\left(z, \chi\left(i\partial_{\phi A}, i\partial_{\phi B}\right)\right), \end{aligned} \tag{38}$$

and introduce operators J_3 and $J_\pm$ (J_3 is called the Casimir operator [19]):

$$\begin{aligned} J_+ &= e^{i(s_A\phi_A + s_B\phi_B)}\mathscr{A}^+, \\ J_- &= \mathscr{A}^- e^{-i(s_A\phi_A + s_B\phi_B)} \\ J_3^A &= \left(k_A - i\frac{\partial_{\phi_A}}{s_A}\right), \\ J_3^B &= \left(k_B - i\frac{\partial_{\phi_B}}{s_B}\right). \end{aligned} \tag{39}$$

According to Equation (39) and $W(x, A, B) = -A \tanh(npx) - B \coth(mpx)$, using the properties $e^{\mp is\phi}J_3 e^{\pm is\phi} = J_3 \pm s$, $e^{\mp is\phi}{J_3}^2 e^{\pm is\phi} = (J_3 \pm s)^2$, we can figure out J_+J_- and J_-J_+:

$$\begin{aligned} J_+J_- &= e^{i(s_A\phi_A + s_B\phi_B)}\mathscr{A}^+\left(z, \chi\left(i\partial_{\phi_A}, i\partial_{\phi_B}\right)\right)\mathscr{A}^- \\ &\quad \cdot \left(z, \chi\left(i\partial_{\phi_A}, i\partial_{\phi_B}\right)\right)e^{-i(s_A\phi_A + s_B\phi_B)} \\ &= -\frac{d^2}{dz^2} + \left[s_A\left(k_A - J_3^A + 1\right)np - s_A^2\left(k_A - J_3^A + 1\right)^2\right]\operatorname{sech}^2(npz) \\ &\quad - \left[s_B\left(k_B - J_3^B + 1\right)mp - s_B^2\left(k_B - J_3^B + 1\right)^2\right]\operatorname{csch}^2(mpz) \\ &\quad + 2s_As_B\left(k_A - J_3^A + 1\right)\left(k_B - J_3^B + 1\right)\coth(mpz)\tanh(npz) \\ &\quad + s_B^2\left(k_B - J_3^B + 1\right)^2 + s_A^2\left(k_A - J_3^A + 1\right)^2, \\ J_-J_+ &= \mathscr{A}^-\left(z, \chi\left(i\partial_{\phi_A}, i\partial_{\phi_B}\right)\right)\mathscr{A}^+\left(z, \chi\left(i\partial_{\phi_A}, i\partial_{\phi_B}\right)\right) \\ &= -\frac{d^2}{dz^2} - \left[s_A\left(k_A - J_3^A\right)np + s_A^2\left(k_A - J_3^A\right)^2\right]\operatorname{sech}^2(npz) \\ &\quad + \left[s_B\left(k_B - J_3^B\right)mp + s_B^2\left(k_B - J_3^B\right)^2\right]\operatorname{csch}^2(mpz) \\ &\quad + 2s_As_B\left(k_A - J_3^A\right)\left(k_B - J_3^B\right)\coth(mpz)\tanh(npz) \\ &\quad + s_B^2\left(k_B - J_3^B\right)^2 + s_A^2\left(k_A - J_3^A\right)^2. \end{aligned} \tag{40}$$

In terms of the requirement of the potential algebra of the shape invariance, J_+J_- and J_-J_+ (in their expressions, the terms containing the independent variable z must cancel each other) will satisfy

$$J_+J_- - J_-J_+ = F\left(J_3^A, J_3^B\right). \tag{41}$$

So, we have

$$s_A = np, s_B = mp, J_3^B + J_3^A = k_A + k_B + 1. \tag{42}$$

According to the reference [21, 22]

$$\begin{aligned} F\left(J_3^A, J_3^B\right) &= G\left(J_3^A, J_3^B\right) - G\left(J_3^A - s_A, J_3^B - s_B\right) \\ &= -\left(s_A^2\left(k_A - J_3^A\right)^2 + s_B^2\left(k_B - J_3^B\right)^2\right) \\ &\quad - \left[-\left(s_B^2\left(k_B - J_3^B + 1\right)^2 + s_A^2\left(k_A - J_3^A + 1\right)^2\right)\right], \end{aligned} \tag{43}$$

so

$$G\left(J_3^A, J_3^B\right) = -\left(s_A^2\left(k_A - J_3^A\right)^2 + s_B^2\left(k_B - J_3^B\right)^2\right). \tag{44}$$

And from

$$\begin{aligned} \mathscr{H}_-\psi_l(z) &= J_+J_-\psi_l(z) = E_l^-\psi_l(z) \\ &= [G(h_A - l - 1, h_B - l - 1) - G(h_A - 1, h_B - 1)]\psi_l(z), \end{aligned} \tag{45}$$

we have

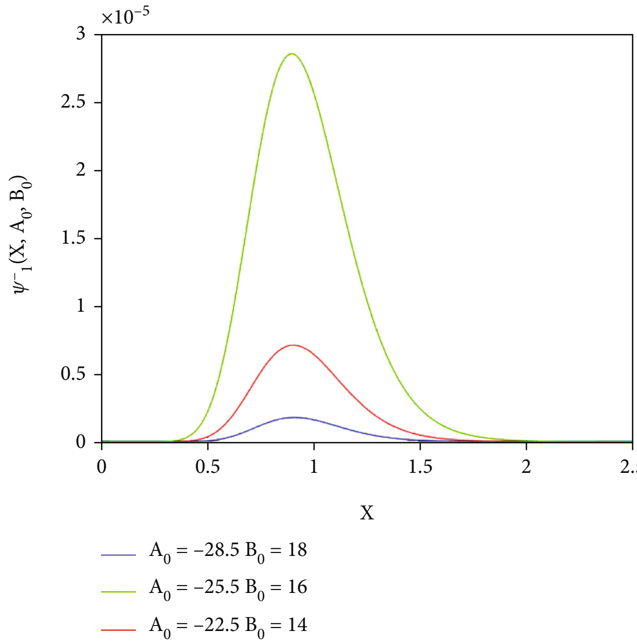

FIGURE 5: The ground wave functions: $\psi_0^-(x, A_0, B_0)$ $(m = 2, n = 3, p = 0.5)$.

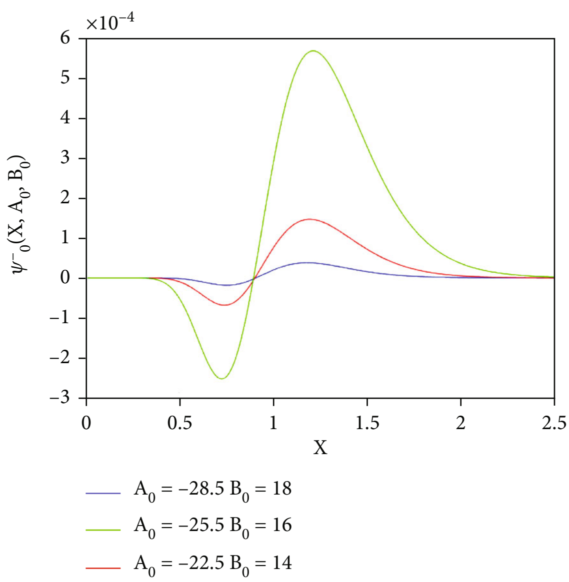

FIGURE 6: The first excited state wave functions: $\psi_1^-(x, A_0, B_0)$ $(m = 2, n = 3, p = 0.5)$.

$$\begin{aligned} E_l^- &= G(h_A - l - 1, h_B - l - 1) - G(h_A - 1, h_B - 1) \\ &= \left(s_A^2 (k_A - h_A + 1)^2 + s_B^2 (k_B - h_B + 1)^2\right) \\ &\quad - \left(s_A^2 (k_A - h_A + 1 + l)^2 + s_B^2 (k_B - h_B + 1 + l)^2\right) \end{aligned} \tag{46}$$

Setting $A = s_A[k_A - (h_A - s_A)]$, $B = s_B[k_B - (h_B - s_B)]$, we get

$$E_l^- = A^2 + B^2 - \left[(A + lnp)^2 + (B + lmp)^2\right]. \tag{47}$$

Comparing Equation (19) with Equation (47), we find that the shape invariance of the partner potential with the generalized hyperbolic Pöschl-Teller superpotential is anastomotic to itself in potential algebraic form. This potential algebraic form corresponds to $A_1 = A_0 + np$ and $B_1 = B_0 + mp$, namely, case I.

For case II, $A_1 = A_0 + np$ and $B_1 = -B_0$, it is a prerequisite for us to carry out our study with a similar substitution of Equation (38). According to Equation (38) and (39), we can figure out J_+J_- and J_-J_+ (when it comes to calculating the equation of J_+J_-, on account that only the parameter A meets the additivity feature, we only need to replace the parameter A with the relevant operator. For the parameter B, we directly replace it as $-B$):

$$\begin{aligned} J_+J_- &= e^{is_A\phi_A}\mathscr{A}^+\left(z, \chi\left(i\partial_{\phi_A}, -B_0\right)\right)\mathscr{A}^-\left(z, \chi\left(i\partial_{\phi_A}, -B_0\right)\right)e^{-is_A\phi_A} \\ &= -\frac{d^2}{dz^2} + \left[s_A\left(k_A - J_3^A + 1\right)np - s_A^2\left(k_A - J_3^A + 1\right)^2\right]\operatorname{sech}^2(npz) \\ &\quad + \left[B_0 mp + B_0^2\right]\operatorname{csch}^2(mpz) - 2s_A B_0\left(k_A - J_3^A + 1\right)\coth(mpz)\tanh(npz) \\ &\quad + B_0^2 + s_A^2\left(k_A - J_3^A + 1\right)^2, \\ J_-J_+ &= \mathscr{A}^-\left(z, \chi\left(i\partial_{\phi_A}, B_0\right)\mathscr{A}^+\left(z, \chi\left(i\partial_{\phi_A}, B_0\right)\right.\right. \\ &= -\frac{d^2}{dz^2} - \left[s_A\left(k_A - J_3^A\right)np + s_A^2\left(k_A - J_3^A\right)^2\right]\operatorname{sech}^2(npz) \\ &\quad + \left[B_0 mp + B_0^2\right]\operatorname{csch}^2(mpz) + 2s_A B_0\left(k_A - J_3^A\right)\coth(mpz)\tanh(npz) \\ &\quad + B_0^2 + s_A^2\left(k_A - J_3^A\right)^2. \end{aligned} \tag{48}$$

In terms of the requirement of the potential algebra of the shape invariance, J_+J_- and J_-J_+ (in their expressions, the terms containing the variable z must cancel each other) will satisfy

Table 2: The summary of the generalized hyperbolic Pöschl-Teller superpotential.

Cases	Expression of eigenvalue	Parameter range	Eigenvalue
I: $A_1 = A_0 + np$ $B_1 = B_0 + \mathrm{m}p$	$E_k^-(A_0, B_0) = -2A_0knp - 2B_0kmp - k^2n^2p^2 - k^2m^2p^2$	$A > 0, B > 0,\ k = 0$	$E_0^- = 0(k = 0)$
		$A < 0, B < 0$ $k < 1 + (B_0/mp) \text{or}\, k < -B_0/mp 0 \le k < 1/2$	$E_0^- = 0(k = 0)$
		$A > 0, B < 0$ $0 \le k < -B_0/mp, 0 \le k < 1/4$	$E_0^- = 0(k = 0)$
		$A > 0, B < 0$ $0 \le k < -(A_0n + B_0m)/p(n^2 + m^2) - 1/20 \le k < 1/2$	$E_0^- = 0(k = 0)$
		$A < 0, B > 0$ $0 \le k < (B_0(n^2 - m^2)/mp(n^2 + m^2)) + n^2/n^2 + m^2 - 1/2$	Depend on the parameters A_0, B_0, n, m, p
II: $A_1 = A_0 + np$ $B_1 = -B_0$	$E_k^-(A_0) = -knp(2A_0 + knp)$ $(B_0 = 0, A_1 = A_0 + np)$	$0 \le k < -A_0/np$	Depend on the parameters A_0, n, p
	$E_k^-(A_0) = -knp(-np + knp)$ $(B_0 = -B_1 \ne 0, A_0 = -np/2)$	$k = 0$	$E_0^- = 0$

$$J_+J_- - J_-J_+ = F\left(J_3^A, J_3^B\right). \tag{49}$$

So, we have

(i) For $B_0 = 0$, we can acquire the particular form of Rosen-Morse (hyperbolic).

(ii) For $B_0 \ne 0$, we should guarantee that the cross term (coth (mpz) tanh (npz)) possesses the same coefficient value, so there is $-2s_AB_0(k_A - J_3^A + s_A) = 2s_AB_0(k_A - J_3^A)$. It is the reason for $F(J_3^A, B_0)$ being equal to zero. That is to say, the case only corresponds to the monomorphism and it is coincident with the discussion in 3.2 case II of this paper

5. Conclusions

Because Pöschl-Teller potential and its generalized potentials are applied widely in atomic and molecular physics and nuclear physics, it is significant to research those potentials further. In this paper, we investigate the solution of the Schrödinger equation with the potential of the generalized superpotential $-A$ tanh $(npx) - B$ coth (mpx). In Section 3, we derive three kinds of additive shape invariances of partner potentials with the generalized superpotential based on the principles of SUSYQM. To guarantee that the supersymmetry is unbroken, we carry out the discussions and figure out eigenfunctions and eigenvalues, respectively, in three cases shown in Table 2. As can be seen from the table, the eigenvalue only exists in the sole zero solution except for two situations. In the two situations, the eigenvalues all rely on the parameter values. That is to say, the number of k is determined when these parameters are specific. For example, in the fourth section of 3.1 case I, by assuming $B_0 = 18$, $A_0 = -28.5$, $m = 2$, $n = 3$, $p = 0.5$, we can acquire the eigenvalues and eigenfunctions. Moreover, in Table 1, Figures 5 and 6, we show the eigenvalues and eigenfunctions, respectively. In the latter part of the article, we analyze the shape invariance again through the potential algebra approach. The shape invariances of the partner potentials are anastomotic in their potential algebraic forms.

Although the shape invariance of the partner potentials with superpotential $-A$ tanh $(npx) - B$ coth (mpx) has two-parameter characteristics, from the present discussion, we can see that it still belongs to the single-parameter case. However, compared with the natural single-parameter shape invariance, that of the generalized hyperbolic Pöschl-Teller superpotential is much more complicated. In addition, the solutions of the Schrödinger equation with such potential deduced from the superpotential are more complex. Starting from the generalization of hyperbolic Pöschl-Teller superpotential in this paper, it also has a more remarkable reference significance for other potentials. In SUSYQM, there are many solvable potentials with two parameters at present, such as the scarf I potential, the scarf II potential, and the Pöschl-Teller I (A coth $(px) - B$ csch (px)). The further study of these potentials can follow the method of this paper: first, make a more general generalization of the superpotentials of these potentials, then study the shape invariance of the partner potentials corresponding with those superpotentials, and then analyze the relationship between shape invariance and parameters. This can not only clarify the influence of two parameters on the shape invariant partner potentials, but also get some new characteristics of shape invariance under two-parameter constraints. Does the shape invariance of these two-parameter partner potentials really show the two-parameter characteristic, and will it bring some new meaningful results? According to the recent results of our study group, a further study of the shape invariance of the two parameters is well expected.

Data Availability

The data availability statement: the data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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