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Analyzing of Nonlinear Generalized Duffing Oscillators Using the Equivalent Linearization Method with a Weighted Averaging

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Authors' contributions

This work was carried out in collaboration between both authors. Author DVH designed the study, performed the statistical analysis, wrote the protocol, wrote the first draft of the manuscript, and managed the analyses of the study. Author NQH managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

The generalized Duffing oscillator is investigated in this paper by using the Equivalent Linearization method with a weighted averaging. Applying of the Equivalent Linearization method in which the averaging value is calculated in a new way called the weighted averaging value by introducing a weighted coefficient function, the amplitude-frequency relationship of the oscillator is obtained in a closed-form. The obtained solutions have been compared with approximate analytical solutions, exact solutions and numerical solutions. Comparisons show the reliability of the present solutions.

Keywords: Equivalent linearization method; weighted averaging; generalized Duffing; nonlinear oscillator.

1 Introduction

There are many problems in physics and engineering that lead to the nonlinear differential Duffing equation: from the oscillation of a simple pendulum, including nonlinear electrical circuits, to various applications in

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image processing. There are many papers published dealing with the nonlinear differential Duffing equation [1,2,3,4,5,6,7,8,9,10]. However, these articles mainly deal with nonlinear Duffing oscillator with third, fifth, or seventh order nonlinear terms. There are very few articles investigating nonlinear Duffing systems with higher order nonlinearities.

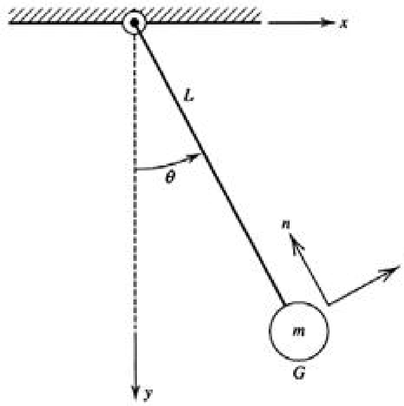

Fig. 1. The simple pendulum

The simplest example of the Duffing equation is the motion of a mathematical pendulum in Fig. 1. When friction is neglected, the differential equation governing the free oscillation of the mathematical pendulum is given by:

$$\ddot{\theta}+\Omega^2\sin\theta=0,\ \ \Omega^2=g/l \tag{1}$$

where g is the gravitational acceleration, l is the length of the pendulum and θ is the deviation angle from the vertical equilibrium position.

Based on Taylor-Maclaurin expansion, the approximation of $sin(\theta)$ is considered:

$$\sin(\theta)=\theta-\frac{1}{6}\theta^3+\frac{1}{120}\theta^5-\frac{1}{5040}\theta^7+\frac{1}{362880}\theta^9-\frac{1}{39916800}\theta^{11}+\frac{1}{6227020800}\theta^{13}+\cdots O(\theta^n) \tag{2}$$

Hence, Eq. (1) can be rewritten as:

$$\ddot{\theta}+\Omega^2\left[\theta-\frac{1}{6}\theta^3+\frac{1}{120}\theta^5-\frac{1}{5040}\theta^7+\frac{1}{362880}\theta^9-\frac{1}{39916800}\theta^{11}+\frac{1}{6227020800}\theta^{13}\right]=0 \tag{3}$$

With $n=13$, Eq. (3) is the nonlinear Duffing oscillator with thirteenth order nonlinearity. If $n=3$, we have the cubic nonlinear Duffing oscillator; and if $n=5$, we have the cubic – quintic nonlinear Duffing oscillator. These oscillators have been studied by many authors [1,2,3,5,6,7,8,9,10]. Several approaches have been proposed so far dealing with the nonlinear Duffing oscillator. He's Frequency - Amplitude Formulation (HFAF) [11,6,12], Iteration Perturbation method (IPM) [13,14], Homotopy Perturbation method (HPM) [5,15], He's Energy Balance method (HEBM) [4,16,17], He's Parameter - Expanding method (HPEM) [18,19] and He's Max - Min approach (HMMA) [20,21,22] are some examples. And recently, a new Amplitude-Frequency Relationship method was introduced by He [23], this method was then applied by Gaxiola to find periodic solution for strongly nonlinear oscillators [24]. Appropriate analytical methods give us an extremely effective tool for analyzing nonlinear oscillations.

The generalized Duffing equation given by in Eq. (4) representing a variety of nonlinear oscillation problems was solved by using HFAF and HEBM for the first time. This work was done by Younesian et al., in 2010 [4]. All odd-type forcing functions can be then involved in the general solution. Natural frequency of the system is obtained as a function of the initial amplitude and the general solution is obtained for any arbitrary power of n. Accuracy and validity of the proposed techniques are then examined by comparing the results obtained based on the HFAF, HEBM with exact integration method. Very good correlations between the three methods are achieved.

$$\ddot{X}+X+\alpha_3X^3+\alpha_5X^5+\alpha_7X^7+\cdots+\alpha_nX^n=0,\ \text{ with: }\ X(0)=A,\ \dot{X}(0)=0 \tag{4}$$

The Equivalent Linearization method (ELM) is one of the common approaches to approximate analysis of dynamical systems. The original linearization for deterministic systems was proposed by Krylov and Bugoliubov [25]. Then Caughey [26] expanded the method for stochastic systems. To date, there have been some extended versions of the Equivalent Linearization method [27,1]. It has been shown that the Equivalent Linearization method is presently the simplest tool widely used for analyzing nonlinear stochastic problems. Nevertheless, the accuracy of the Equivalent Linearization method with conventional averaging normally reduces for middle or strong nonlinear systems. A reason is that some terms will vanish in the averaging process, for example, the averaging values of the functions sin(t) and cos(t) over one period will be equal to zero, this makes information related to these quantities will be lost, and that often leads to large errors in the solution of problem. In 2015, Anh [2] proposed a new way for determining averaging values, instead of using conventional averaging value, author introduced weighted coefficient functions, thus the averaging values was given in a new way called the weighted averaging values. And the proposed method have been applied very effectively in analyzing of some strongly nonlinear oscillators [3].

In this paper, Anh's method will be applied to analyse the generalized Duffing oscillator given in Eq. (4). The obtained solutions are compared with the solutions published by Younesian et al. [4]. Comparisons show the accuracy of the present solutions.

2 Solution Procedure

Using the Equivalent Linearization method we will find the approximate solution of Eq. (4). First, Eq. (4) is replaced by the linearized equation as follows:

$$\ddot{X}+\omega^2X=0 \tag{5}$$

where the cofficient ω^2 of the linear term is determined from the mean square error criterion which requires the mean square of equation error to be minimum:

$$\left\langle e^2(X)\right\rangle=\left\langle\left(X+\alpha_3X^3+\alpha_5X^5+\alpha_7X^7+\cdots+\alpha_nX^n-\omega^2X\right)^2\right\rangle\to \underset{\omega^2}{Min} \tag{6}$$

Thus, from

$$\frac{\partial\left\langle e^2(X)\right\rangle}{\partial\omega^2}=0$$

yields:

$$\omega=\sqrt{1+\alpha_3\frac{\left\langle X^4\right\rangle}{\left\langle X^2\right\rangle}+\alpha_5\frac{\left\langle X^6\right\rangle}{\left\langle X^2\right\rangle}+\alpha_7\frac{\left\langle X^7\right\rangle}{\left\langle X^2\right\rangle}+\cdots+\alpha_n\frac{\left\langle X^{n+1}\right\rangle}{\left\langle X^2\right\rangle}} \tag{7}$$

ω given in Eq. (7) is the approximate frequency of the generalized nonlinear Duffing oscillator given by Eq. (4).

In Eq. (7), the symbol $\langle\bullet\rangle$ denotes the time-averaging operator in classical (convential) meaning [25]:

$$\langle X(t)\rangle = \lim_{T\to+\infty}\frac{1}{T}\int_0^T X(t)dt \tag{8}$$

For a ω-frequency function $X(\omega t)$, the averaging process is taken during one period T, i.e.:

$$\langle X(\omega t)\rangle = \frac{1}{T}\int_0^T X(\omega t)dt = \frac{1}{2\pi}\int_0^{2\pi} X(\tau)d\tau, \quad \tau=\omega t \tag{9}$$

The averaging values in Eqs. (8) and (9) are called the classical or conventional averaging values. They often lead to inaccurate results, especially for some periodic functions such as *sine* or *cosine* ones.

In this paper, the weighted averaging value proposed by Anh [2] is used to calculate averaging values in Eq. (7) in stead of the conventional averaging values in Eq. (8) or (9). The idea of the proposed method as follows: replacing the constant coefficient *1/T* in Eqs.(8) and (9) by a weighted coefficient function *h(t)*. Thus we get so-called a weighted average value:

$$\langle X(t)\rangle = \int_0^T h(t)X(t)dt \tag{10}$$

with the condition:

$$\int_0^T h(t)dt = 1 \tag{11}$$

In Ref. [2], Anh has proposed a weighted coefficient function as follows:

$$h(t) = s^2\omega t\, e^{-s\omega t} \tag{12}$$

where s is positive constant.

It is clear that the weighted coefficient function (12) satisfies the condition (11). The weighted coefficient function (12) is obtained as a product of the optimistic weighted coefficient t and the pessimistic weighted coefficient $e^{-s\omega t}$, which has one maximal value at $t_{\max} = 1/(\omega s)$ and then decreases to zero as $t\to+\infty$. For case $s=0$, the weighted coefficient function (12) has maximal value at infinity, this case corresponds to the classical or conventional averaging value. The detailed properties of the weighted coefficient function *h(t)* in Eq. (12) can be viewed in Refs. [2,3].

The solution of the linearized equation (5) is given by:

$$X(t) = A\cos(\omega t) \tag{13}$$

With the periodic solution of linearized equation (5) in Eq. (13), the averaging values in Eq. (7) can be calculated by using Eq. (10) with the weighted coefficient function given in Eq. (12) and Laplace tranform, for example:

$$\langle X^2\rangle = \langle A^2\cos^2(\omega t)\rangle = \int_0^{+\infty} A^2 s^2\omega t\, e^{-s\omega t}\cos^2(\omega t)dt = \int_0^{+\infty} A^2 s^2 \tau\, e^{-s\tau}\cos^2(\tau)d\tau = A^2\frac{s^4+2s^2+8}{(s^2+4)^2} \tag{14}$$

It is similar, we get:

$$\langle X^4\rangle = \langle A^4\cos^4(\omega t)\rangle = A^4\frac{s^8+28s^6+248s^4+416s^2+1536}{(s^2+4)^2(s^2+16)^2} \tag{15}$$

$$\langle X^6\rangle = A^6\frac{s^{12}+94s^{10}+3168s^8+45712s^6+282496s^4+440064s^2+1658880}{(s^2+4)^2(s^2+16)^2(s^2+36)^2} \tag{16}$$

$$\left\langle X^8 \right\rangle = A^8 \frac{\begin{pmatrix} s^{16} + 2165s^{14} + 18256s^{12} + 768000s^{10} + 17013120s^8 + \\ 192596992s^6 + 1014806528s^4 + 1516142592s^2 + 5945425920 \end{pmatrix}}{(s^2+4)^2(s^2+16)^2(s^2+36)^2(s^2+64)^2} \quad (17)$$

$$\left\langle X^{10} \right\rangle = A^{10} \frac{\begin{pmatrix} 330883456s^{12} + 6277280s^{14} + 13076221132800s^2 + \\ 53508833280000 + 8994366406656s^4 + 1872317542400s^6 + \\ 188571698176s^8 + 10357551360s^{10} + 69336s^{16} + 410s^{18} + s^{20} \end{pmatrix}}{(s^2+4)^2(s^2+16)^2(s^2+36)^2(s^2+64)^2(s^2+100)^2} \quad (18)$$

For $n = 11, 13, 15, \ldots$, the coresponding averaging values $\left\langle X^{12} \right\rangle, \left\langle X^{14} \right\rangle, \left\langle X^{16} \right\rangle, \ldots$ can be calculated in the same manner.

Substituting the above averaging values calculated by using the proposed method into Eq. (7), we get the approximate frequency of the generalized nonlinear Duffing oscillator given in Eq. (4); and from Eq. (13) we get the approximate solution of the oscillator.

3 Numerical Results

In this section, we present and discuss the numerical results obtained by employing the present method with the results obtained by employing He's Energy Balance method (HEBM), exact integration method and Runge – Kutta method. And in this paper, the parameter *s* in the expression of weighted coefficient function *h(t)* is chosen equal to 2.

We note that the approximate frequency of the generalized Duffing oscillator (4) was found by employing HEBM as follows [4]:

$$\omega_{HEBM} = \sqrt{1 + 4 \sum_{\substack{n=2k+1 \\ k=1,2,3,\ldots m}}^{n} \frac{\alpha_n A^{n-1}}{n+1} \left[1 - \left(\frac{\sqrt{2}}{2} \right)^{n+1} \right]} \quad (19)$$

3.1 Example 1, for *n=3*

The cubic nonlinear Duffing oscillator has the following form in this case:

$$\ddot{X} + X + \alpha_3 X^3 = 0 \quad (20)$$

The approximate frequency obtained by the present method is derived from Eq. (7), as follows (note that *s=2*):

$$\omega_{present} = \sqrt{1 + 0.72\alpha_3 A^2} \quad (21)$$

The approximately frequency obtained by using HEBM can be derived from Eq. (19):

$$\omega_{HEBM} = \sqrt{1 + \frac{3}{4}\alpha_3 A^2} \quad (22)$$

Accuracy of this method for this example is shown in Table 1 and Figs. 2–4. Table 1 shows the comparison between the approximate frequencies $\omega_{present}$ obtained by this method, the ones obtained by HEBM ω_{HEBM} [4] and the exact ones ω_e [7], the relative error is obtained for small and large values of the initial amplitude *A*. Table 1 shows that the maximum relative error is less than 0.156% for this method and

2.22% for HEBM, respectively. Comparions of trend the relative error of this case for two methods are illustrated in Figs. 2-3 for a range of initial amplitudes A and parameters α_3, respectively. Again, the accuracy of the present method can be seen. The validity of the solution technique is guaranteed for strong nonlinearities.

The exact frequency of the oscillator given by Eq. (20) is [7]:

$$\omega_e = \frac{2\pi}{4\sqrt{2}\int_0^{\pi/2} \frac{dt}{\sqrt{\alpha_3 A^2 \cos^2(t) + \alpha_3 A^2 + 2}}} \tag{23}$$

Table 1. Comparison of the approximate frequencies with the exact frequency, *n=3*

A	α_3	ω_e	ω_{HEBM}	R. Error (%)	$\omega_{present}$	R. Error (%)
0.01	0.1	1.000004	1.000004	**0.000000**	1.000004	**0.000000**
1	0.1	1.036717	1.036822	**0.010128**	1.035374	**0.129544**
10	0.1	2.866640	2.915476	**1.703597**	2.863564	**0.107303**
100	0.1	26.810738	27.404379	**2.214191**	26.851443	**0.151823**
0.01	1	1.000037	1.000037	**0.000000**	1.000036	**0.000100**
1	1	1.317776	1.322876	**0.387016**	1.311488	**0.477168**
10	1	8.533586	8.717798	**2.158670**	8.544004	**0.122082**
100	1	84.727479	86.608314	**2.219864**	84.858707	**0.154883**
0.01	10	1.000375	1.000375	**0.000000**	1.000359	**0.001599**
1	10	2.866640	2.915476	**1.703597**	2.863564	**0.107303**
10	10	26.810738	27.404379	**2.214191**	26.851443	**0.151824**
100	10	267.914253	273.863104	**2.220431**	268.330021	**0.155187**
0.01	100	1.003742	1.003743	**0.000099**	1.003594	**0.014745**
1	100	8.533586	8.717798	**2.158670**	8.544004	**0.122082**
10	100	84.727479	86.608314	**2.219864**	84.858706	**0.154881**
100	100	847.213702	866.025981	**2.220488**	848.528727	**0.155218**

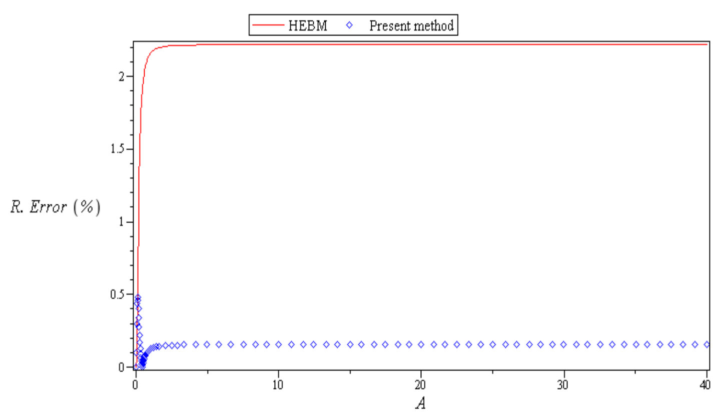

Fig. 2. Comparison of the relative errors of approximate frequencies obtained by two methods for *n=3*, α_3*=100*

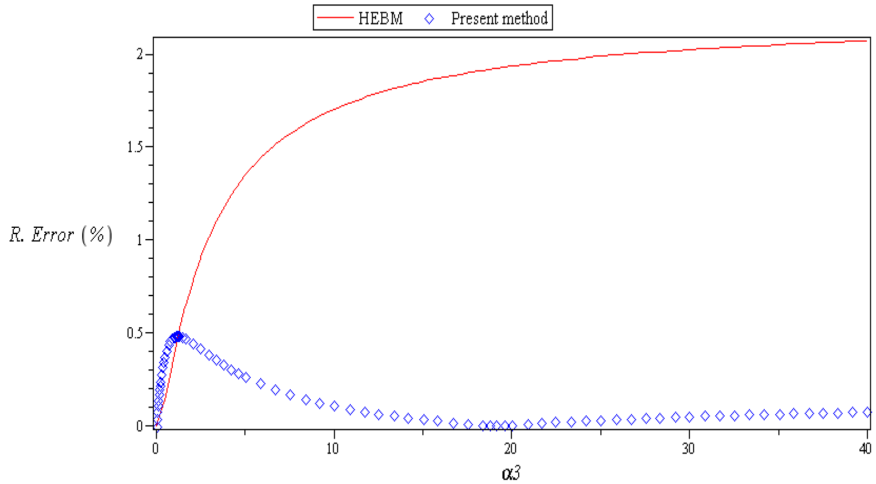

Fig. 3. Comparison of the relative errors of approximate frequencies obtained by two methods for *n=3*, *A=1.*

Comparisons of the responses of the present solution and the HEBM solution are given in Fig. 4. Numerical results validate again the accuracy of the present method.

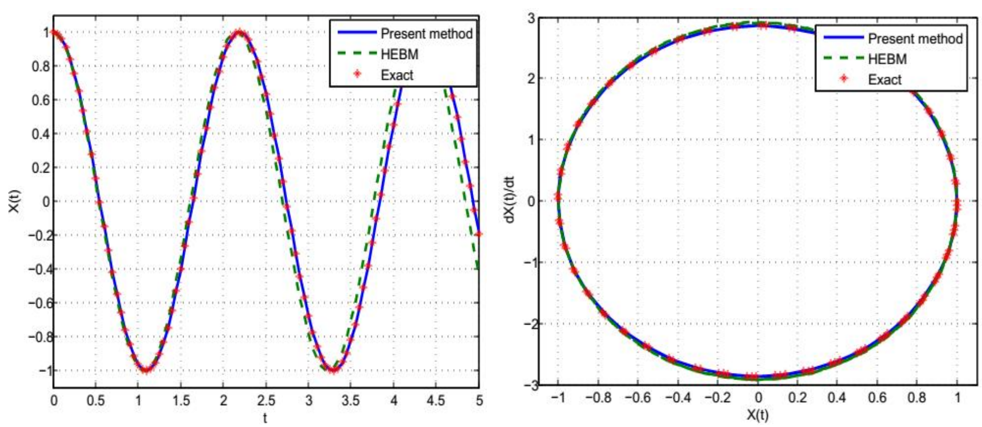

Fig. 4. Comparison of the HEBM and present solutions with the exact solution for *n=3*, *A=1*, α_3*=10.*

3.2 Example 2, for *n=5*

In this case, the cubic-quintic nonlinear Duffing oscillator has the following form:

$$\ddot{X} + X + \alpha_3 X^3 + \alpha_5 X^5 = 0 \tag{24}$$

From Eq. (7), the approximate frequency for this case obtained by the present method as follows:

$$\omega_{present} = \sqrt{1 + 0.72\alpha_3 A^2 + 0.575\alpha_5 A^4} \tag{25}$$

The approximately frequency obtained by using HEBM for this case can be derived from Eq. (19):

$$\omega_{HEBM} = \sqrt{1 + \frac{3}{4}\alpha_3 A^2 + \frac{7}{12}\alpha_5 A^4} \tag{26}$$

To show the accuracy of the present solution, comparison of the approximate frequencies obtained by the two methods are showed in Table 2 for $\alpha_3=10$ and $\alpha_5=100$. It can be seen from Table 2 that the maximum relative error of the present frequency is only 1.67% while the maximum relative error of the HEBM freqency reaches to 2.27%.

Table 2. Comparison of the approximate frequencies with the exact frequency, *n*=5

A	ω_e	ω_{HEBM}	R. error (%)	$\omega_{present}$	R. error (%)
0.1	1.039700	1.038188	**0.145427**	1.039591	**0.010484**
0.5	2.524694	2.538865	**0.561296**	2.543374	**0.739892**
1	8.010045	8.156797	**1.832100**	8.124038	**1.423126**
5	187.199101	191.414036	**2.251579**	190.068409	**1.532757**
10	747.323025	764.234475	**2.262937**	758.782578	**1.533414**
100	74683.90825	76376.73294	**2.266653**	75929.24898	**1.667482**

The exact frequency of the oscillator given by Eq. (24) is [8]:

$$\omega_e = \frac{\pi k_1}{2\int_0^{\pi/2}\left(1+k_2\sin^2 t+k_3\sin^4 t\right)^{-1/2}dt} \tag{27}$$

where:

$$k_1 = \sqrt{1+\frac{\alpha_3 A^2}{2}+\frac{\alpha_5 A^4}{3}} \tag{28}$$

$$k_2 = \frac{3\alpha_3 A^2+2\alpha_5 A^4}{6+3\alpha_3 A^2+2\alpha_5 A^4} \tag{29}$$

$$k_3 = \frac{2\alpha_5 A^4}{6+3\alpha_3 A^2+2\alpha_5 A^4} \tag{30}$$

Comparions of trend the relative error of this case between two methods are illustrated in Figs. 5 – 7 for a range of initial amplitudes *A*, parameters α_3 and α_5, respectively. Again, the accuracy of the present solution can be seen for strong nonlinearities.

Comparisons of the responses of the present solution and the HEBM solution are given in Fig. 8.

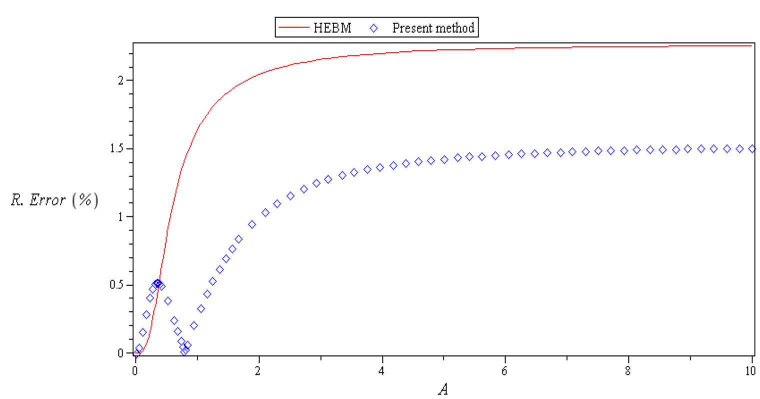

Fig. 5. Comparison of the relative errors of approximate frequencies obtained by two methods for *n*=5, $\alpha_3=10$, $\alpha_5=10$.

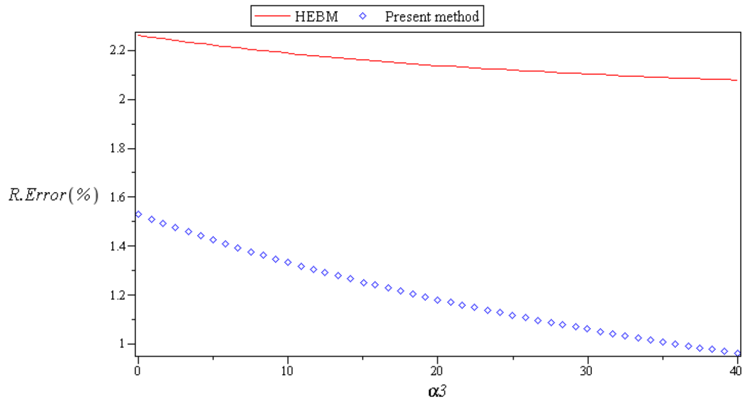

Fig. 6. Comparison of the relative errors of approximate frequencies obtained by two methods for *n*=5, α_5=5, *A*=5.

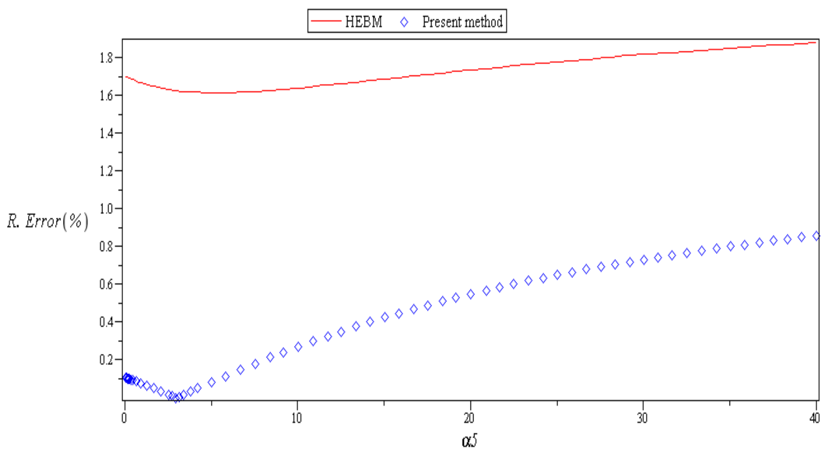

Fig. 7. Comparison of the relative errors of approximate frequencies obtained by two methods for *n*=5, α_3=10, *A*=1.

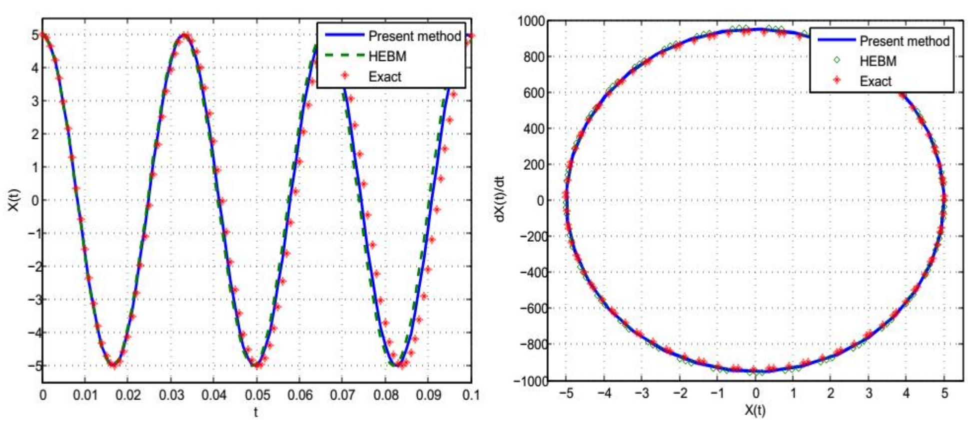

Fig. 8. Comparison of the HEBM and present solutions with the exact solution for *n*=5, *A*=5, α_3=10, α_5=100.

3.3 Example 3, for *n=7*

In this case, the Duffing oscillator has the following form:

$$\ddot{X} + X + \alpha_3 X^3 + \alpha_5 X^5 + \alpha_7 X^7 = 0 \quad (31)$$

The approximate frequency for this case obtained by the present method is derived from Eq. (7), as follows:

$$\omega_{present} = \sqrt{1 + 0.72\alpha_3 A^2 + 0.575\alpha_5 A^4 + 0.436\alpha_7 A^6} \quad (32)$$

Using the HEBM, the approximate frequency can achieve [4]:

$$\omega_{HEBM} = \sqrt{1 + \frac{3}{4}\alpha_3 A^2 + \frac{7}{12}\alpha_5 A^4 + \frac{15}{32}\alpha_7 A^6} \quad (33)$$

Comparison of the approximate frequencies obtained by the two methods are illustrated in Table 3 for $\alpha_3=10$ and $\alpha_5=100$. It can be seen from Table 3 that the maximum relative error of the present frequency is only 0.9% while the maximum relative error of the HEBM freqency reaches to 1.6%.

Table 3. Comparison of the approximate frequencies with the exact frequency, *n=7*

α_3	α_5	α_7	***A***	ω_e	ω_{HEBM}	**R. Error (%)**	$\omega_{present}$	**R. Error (%)**
1	1	1	0.1	1.003773	1.003772	**0.00012**	1.003622	**0.015020**
5	5	5	0.1	1.018704	1.018721	**0.001698**	1.017983	**0.070721**
5	5	5	0.5	1.463311	1.468473	**0.352741**	1.455152	**0.557625**
10	10	10	0.5	1.806022	1.820117	**0.780467**	1.798592	**0.411399**
10	10	10	1	4.305981	4.361288	**1.284413**	4.334240	**0.656274**
50	50	50	1	9.399149	9.544850	**1.550147**	9.483048	**0.892619**

Comparisons of the response of the present solution and the HEBM solution are showed in Fig. 9.

For this oscillator given in Eq. (31), the exact frequency as follows [4]:

$$\omega_e = 2\pi \left[4 \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + \frac{1}{2}(1+\sin^2\theta)\alpha_3 A^2 + \frac{1}{3}(1+\sin^2\theta+\sin^4\theta)\alpha_5 A^4 + \frac{1}{4}(1+\sin^2\theta+\sin^4\theta+\sin^6\theta)\alpha_7 A^6}} \right]^{-1} \quad (34)$$

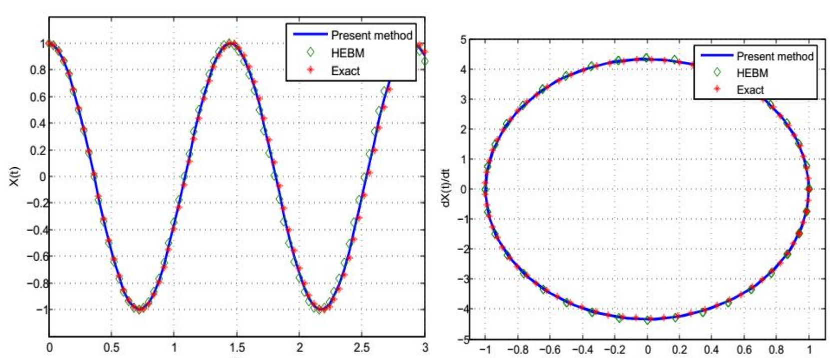

Fig. 9. Comparison of the HEBM and present solutions with the exact solution for *n=7*, *A=1*, $\alpha_3=10$, $\alpha_5=10$, $\alpha_7=10$.

3.4 Example 4, for the higher value of *n*

3.4.1 For *n=9*

The approximate frequency for this case obtained by the present method is derived from Eq. (7), as follows:

$$\omega_{present} = \sqrt{1 + 0.72\alpha_3 A^2 + 0.575\alpha_5 A^4 + 0.436\alpha_7 A^6 + 0.4198\alpha_9 A^8} \tag{35}$$

The approximate HEBM frequency [4]:

$$\omega_{HEBM} = \sqrt{1 + \frac{3}{4}\alpha_3 A^2 + \frac{7}{12}\alpha_5 A^4 + \frac{15}{32}\alpha_7 A^6 + \frac{31}{80}\alpha_9 A^8} \tag{36}$$

3.4.2 For *n=11*

The approximate frequency for this case obtained by the present method is derived from Eq. (8), as follows:

$$\omega_{present} = \sqrt{1 + 0.72\alpha_3 A^2 + 0.575\alpha_5 A^4 + 0.436\alpha_7 A^6 + 0.4198\alpha_9 A^8 + 0.3722\alpha_{11} A^{10}} \tag{37}$$

The approximate HEBM frequency [4]:

$$\omega_{HEBM} = \sqrt{1 + \frac{3}{4}\alpha_3 A^2 + \frac{7}{12}\alpha_5 A^4 + \frac{15}{32}\alpha_7 A^6 + \frac{31}{80}\alpha_9 A^8 + \frac{63}{192}\alpha_{11} A^{10}} \tag{38}$$

For *n=13,15,…* ,the approximate frequency can be get by the same manner.

Comparison of the solutions achived by two methods with the numerical solutions are illustrated in Figs. 10-12 for *n=9*, *n=11* and *n=13*, respectively. It can be seen that there is a great agreement between the analytical solutions and the exact solutions as well as the numerical solutions.

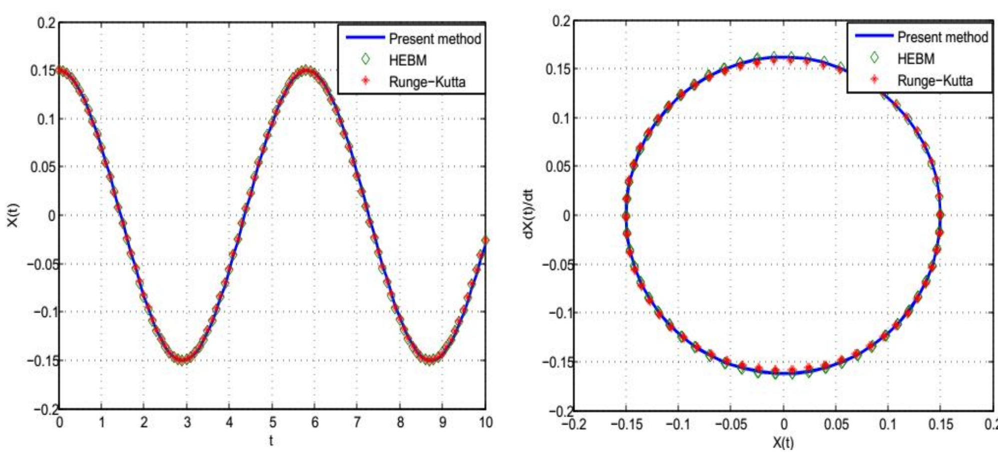

Fig. 10. Comparison of the HEBM and present solutions with the exact solution for *n=9, A=0.5, α₃=10, α₅=10, α₇=10, α₉=5.*

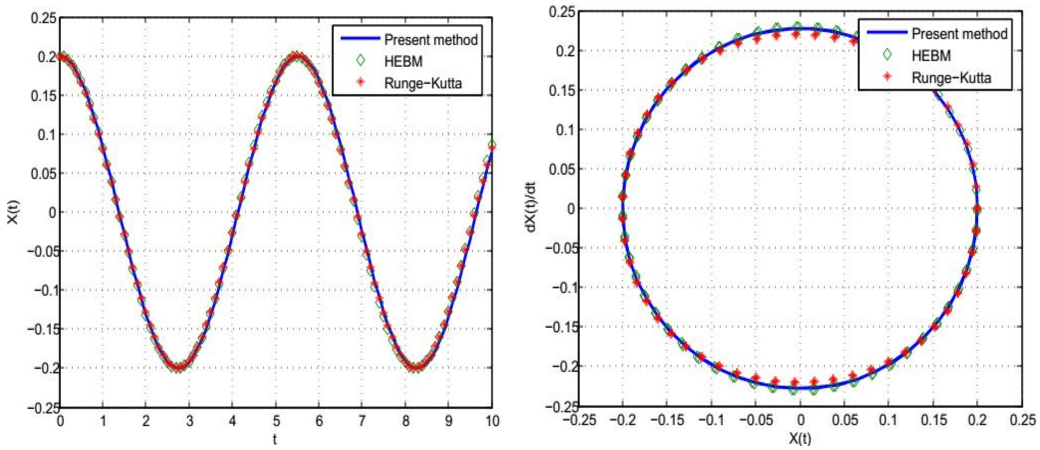

Fig. 11. Comparison of the HEBM and present solutions with the exact solution for $n=11$, $A=0.5$, $\alpha_3=10$, $\alpha_5=10$, $\alpha_7=10$, $\alpha_9=5$, $\alpha_{11}=10$.

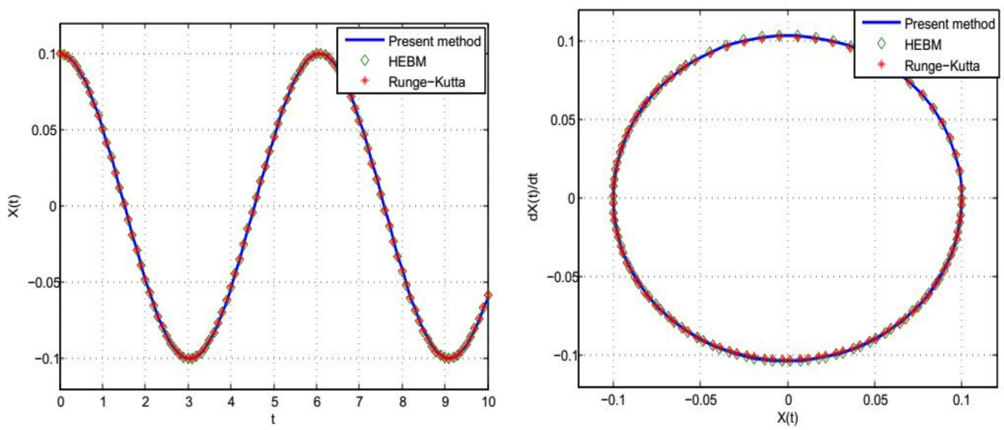

Fig. 12. Comparison of the HEBM and present solutions with the exact solution for $n=13$, $A=0.1$, $\alpha_3=10$, $\alpha_5=10$, $\alpha_7=10$, $\alpha_9=5$, $\alpha_{11}=10$, $\alpha_{13}=20$.

4 Conclusion

Analytical approximation solutions of the generalized Duffing oscillator is investigated by the Equivalent Linearization method with a weighted averaging. The method is developed based on the convenience of the classical Equivalent Linearization method and the accuracy of the weighted averaging value. A new approximate solution for the generalized Duffing is given in this work. The relationship between the frequency and the initial amplitude is given in a closed form. The approximate solutions are the harmonic oscillations, which are compared with the previous analytical solutions, the exact solutions and the numerical solutions. Comparisons show the accuracy of the present solutions.

The solution procedure shows that this method is very simple and it can be further developed for many strongly nonlinear systems in engineering and multi-degree of fredom vibrations.

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Competing Interests

Authors have declared that no competing interests exist.

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